4.33.

$$
\begin{aligned}
y[n] &= x^2[n] \\
Y(e^{j\omega}) &= X(e^{j\omega}) * X(e^{j\omega})
\end{aligned}
$$

therefore, $Y(e^{j\omega})$ will occupy twice the frequency band that $X(e^{j\omega})$ does if no aliasing occurs.
If $Y(e^{j\omega}) \neq 0, \quad -\pi < \omega < \pi$, then $X(e^{j\omega}) \neq 0, \quad -\frac{\pi}{2} < \omega < \frac{\pi}{2}$ and so $X(j\Omega) = 0, \quad |\Omega| \geq 2\pi(1000)$.
Since $\omega = \Omega T$,

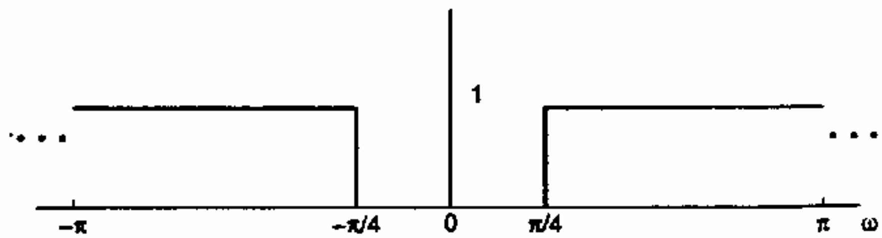
$$
\begin{aligned}
\frac{\pi}{2} &\geq T \cdot 2\pi(1000) \\
T &\leq \frac{1}{4000}
\end{aligned}
$$

4.36. (a) Since $\Omega T = \omega, \quad (2\pi \cdot 100)T = \frac{\pi}{2} \Rightarrow T = \frac{1}{400}$

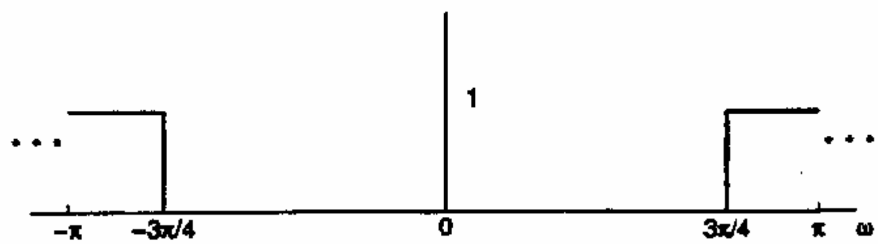
(b) The downsampler has $M = 2$. Since $x[n]$ is bandlimited to $\frac{\pi}{M}$, there will be no aliasing. The frequency axis simply expands by a factor of 2.
For $y_c(t) = x_c(t) \Leftrightarrow Y_c(j\Omega) = X_c(j\Omega)$.
Therefore $\Omega T' \Rightarrow 2\pi \cdot 100T' \Rightarrow T' = \frac{1}{200}$.

5.21. $h_{lp}[n]$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$

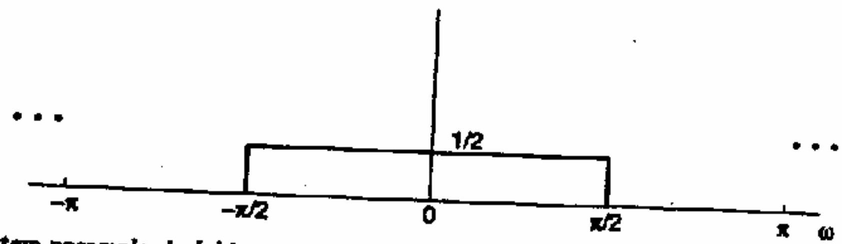
(a) $y[n] = x[n] - x[n] * h_{lp}[n] \Rightarrow H(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$
This is a highpass filter.



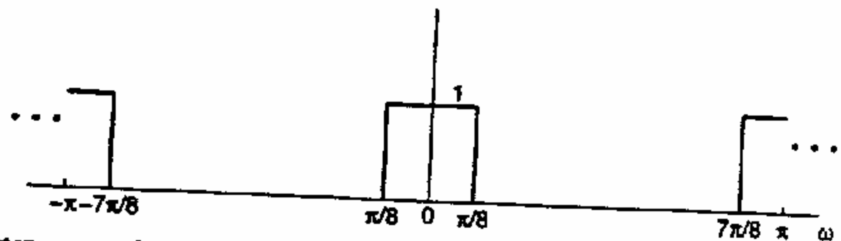
(b) $x[n]$ is first modulated by π, lowpass filtered, and demodulated by π. Therefore, $H_{lp}(e^{j\omega})$ filters the high frequency components of $X(e^{j\omega})$.
This is a highpass filter.



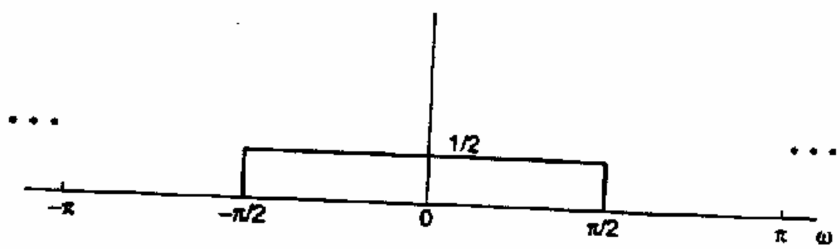
(c) $h_{lp}[2n]$ is a downsampled version of the filter. Therefore, the frequency response will be "spread out" by a factor of two, with a gain of $\frac{1}{2}$.
This is a lowpass filter.



(d) This system upsamples $h_{lp}[n]$ by a factor of two. Therefore, the frequency axis will be compressed by a factor of two.
This is a bandstop filter.



(e) This system upsamples the input before passing it through $h_{lp}[n]$. This effectively doubles the frequency bandwidth of $H_{lp}(e^{j\omega})$.
This is a lowpass filter.



5.22.

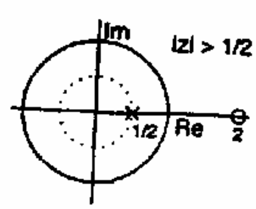
$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}} = \frac{Y(z)}{X(z)}, \quad \text{causal, so ROC is } |z| > a$$

(a) Cross multiplying and taking the inverse transform

$$y[n] - ay[n-1] = x[n] - \frac{1}{a}x[n-1]$$

(b) Since $H(z)$ is causal, we know that the ROC is $|z| > a$. For stability, the ROC must include the unit circle. So, $H(z)$ is stable for $|a| < 1$.

(c) $a = \frac{1}{2}$



(d)

$$H(z) = \frac{1}{1-az^{-1}} - \frac{a^{-1}z^{-1}}{1-az^{-1}}, \quad |z| > a$$

$$h[n] = (a)^n u[n] - \frac{1}{a}(a)^{n-1}u[n-1]$$

(e)

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1-a^{-1}e^{-j\omega}}{1-ae^{-j\omega}}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{1-a^{-1}e^{-j\omega}}{1-ae^{-j\omega}} \cdot \frac{1-a^{-1}e^{j\omega}}{1-ae^{j\omega}}$$

$$\begin{aligned} |H(e^{j\omega})| &= \left(\frac{1+\frac{1}{a^2}-\frac{2}{a}\cos\omega}{1+a^2-2a\cos\omega}\right)^{\frac{1}{2}} \\ &= \frac{1}{a}\left(\frac{a^2+1-2a\cos\omega}{1+a^2-2a\cos\omega}\right)^{\frac{1}{2}} \\ &= \frac{1}{a} \end{aligned}$$

5.24. (a) Taking the z-transform of both sides and rearranging

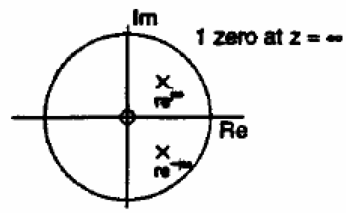
$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{4}+z^{-2}}{1-\frac{1}{4}z^{-2}}$$

Since the poles and zeros {2 poles at $z = \pm 1/2$, 2 zeros at $z = \pm 2$} occur in conjugate reciprocal pairs the system is allpass. This property is easy to recognize since, as in the system above, the coefficients of the numerator and denominator z-polynomials get reversed (and in general conjugated).

(b) It is a property of allpass systems that the output energy is equal to the input energy. Here is the proof.

$$\begin{aligned} \sum_{n=0}^{N-1} |y[n]|^2 &= \sum_{n=-\infty}^{\infty} |y[n]|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Y(e^{j\omega})\right|^2 d\omega && \text{(by Parseval's Theorem)} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})X(e^{j\omega})\right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega && (\left|H(e^{j\omega})\right|^2 = 1 \text{ since } h[n] \text{ is allpass}) \\ &= \sum_{n=-\infty}^{\infty} |x[n]|^2 && \text{(by Parseval's theorem)} \\ &= \sum_{n=0}^{N-1} |x[n]|^2 \\ &= 5 \end{aligned}$$

5.26. (a) A labeled pole-zero diagram appears below.



The table of common z-transform pairs gives us

$$(r^n \sin \omega_0 n)u[n] \longleftrightarrow \frac{(r \sin \omega_0)z^{-1}}{1 - (2r \cos \omega_0)z^{-1} + r^2 z^{-2}}, \quad |z| > r$$

which enables us to derive $h[n]$.

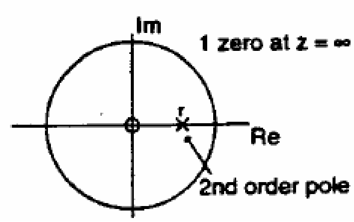
$$h[n] = \left(\frac{1}{\sin \omega_0}\right)(r^n \sin \omega_0 n)u[n]$$

(b) When $\omega_0 = 0$

$$H(z) = \frac{rz^{-1}}{1 - (2r \cos \omega_0)z^{-1} + r^2 z^{-2}} = \frac{rz^{-1}}{(1 - rz^{-1})^2}, \quad |z| > r$$

Again, using a table lookup gives us

$$h[n] = nr^n u[n]$$



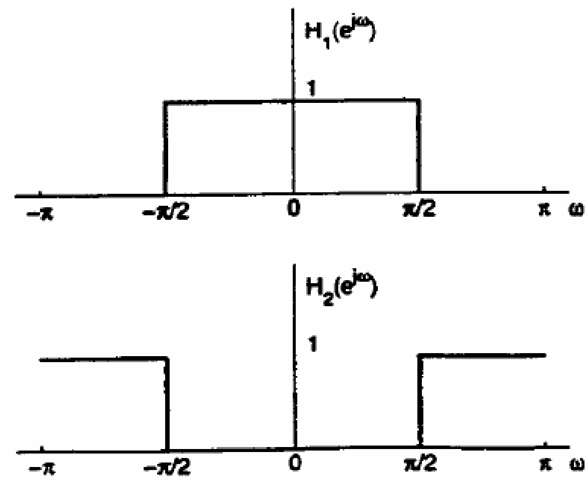
5.27. Making use of some DTFT properties can aide in the solution of this problem. First, note that

$$h_2[n] = (-1)^n h_1[n]$$
$$h_2[n] = e^{-j\pi n} h_1[n]$$

Using the DTFT property that states that modulation in the time domain corresponds to a shift in the frequency domain,

$$H_2(e^{j\omega}) = H_1(e^{j(\omega+\pi)})$$

Consequently, $H_2(e^{j\omega})$ is simply $H_1(e^{j\omega})$ shifted by π. The ideal low pass filter has now become the ideal high pass filter, as shown below.



5.29.

$$H(z) = \frac{21}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})(1-4z^{-1})}$$
$$= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}} + \frac{48}{1-4z^{-1}}$$

Since we know the sequence is not stable, the ROC must not include $|z| = 1$, and since it is two-sided, the ROC must be a ring. This leaves only one possible choice: the ROC is $2 < |z| < 4$.

(a)

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 28(2)^n u[n] - 48(4)^n u[-n-1]$$

(b)

$$H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}}$$

$$H_2(z) = \frac{48}{1-4z^{-1}}$$