2.21. For an arbitrary linear system, we have

$$y[n] = T\{x[n]\},$$

Let $x[n] = 0$ for all n.

$$y[n] = T\{x[n]\}$$

For some arbitrary $x_1[n]$, we have

$$y_1[n] = T\{x_1[n]\}$$

Using the linearity of the system:

$$\begin{aligned} T\{x[n] + x_1[n]\} &= T\{x[n]\} + T\{x_1[n]\} \\ &= y[n] + y_1[n] \end{aligned}$$

Since $x[n]$ is zero for all n,

$$T\{x[n] + x_1[n]\} = T\{x_1[n]\} = y_1[n]$$

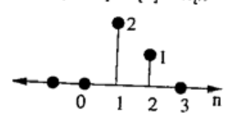
Hence, $y[n]$ must also be zero for all n.

2.22. We use the graphical approach to compute the convolution:

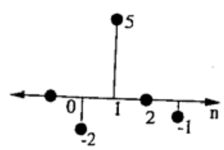
$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

(a) $y[n] = x[n] * h[n]$

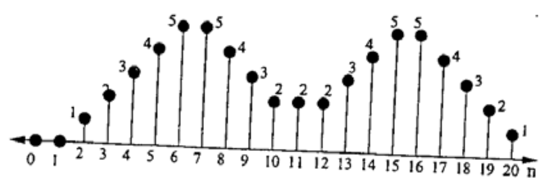
$$y[n] = \delta[n-1] * h[n] = h[n-1]$$



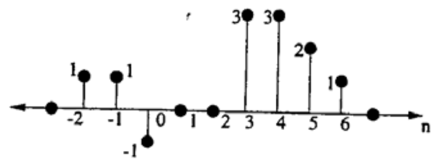
(b) $y[n] = x[n] * h[n]$



(c) $y[n] = x[n] * h[n]$



(d) $y[n] = x[n] * h[n]$



2.24. The response of the system to a delayed step:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} u[k-4]h[n-k] \end{aligned}$$

$$y[n] = \sum_{k=4}^{\infty} h[n-k]$$

Evaluating the above summation:

For $n < 4$: $y[n] = 0$
For $n = 4$: $y[n] = h[0] = 1$
For $n = 5$: $y[n] = h[1] + h[0] = 2$
For $n = 6$: $y[n] = h[2] + h[1] + h[0] = 3$
For $n = 7$: $y[n] = h[3] + h[2] + h[1] + h[0] = 4$
For $n = 8$: $y[n] = h[4] + h[3] + h[2] + h[1] + h[0] = 2$
For $n \geq 9$: $y[n] = h[5] + h[4] + h[3] + h[2] + h[1] + h[0] = 0$

2.25. The output is obtained from the convolution sum:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k]u[n-k] \end{aligned}$$

The convolution may be broken into five regions over the range of n:

$$y[n] = 0, \text{for } n < 0$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{n} a^k \\ &= \frac{1-a^{(n+1)}}{1-a}, \text{for } 0 \leq n \leq N_1 \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k \\ &= \frac{1-a^{(N_1+1)}}{1-a}, \text{ for } N_1 < n < N_2 \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{n} a^{(k-N_2)} \\ &= \frac{1-a^{(N_1+1)}}{1-a} + \frac{1-a^{(n+1)}}{1-a} \\ &= \frac{2-a^{(N_1+1)}-a^{(n+1)}}{1-a}, \text{ for } N_2 \leq n \leq (N_1+N_2) \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{N_1+N_2} a^{(k-N_2)} \\ &= \sum_{k=0}^{N_1} a^k + \sum_{m=0} N_1 a^m \\ &= 2\sum_{k=0}^{N_1} a^k \\ &= 2 \cdot \left(\frac{1-a^{(N_1+1)}}{1-a}\right), \text{ for } n > (N_1+N_2) \end{aligned}$$

2.35. (a) Notice that $x_1[n] = x_2[n] + x_3[n+4]$, so if $T\{\cdot\}$ is linear,

$$\begin{aligned} T\{x_1[n]\} &= T\{x_2[n]\} + T\{x_3[n+4]\} \\ &= y_2[n] + y_3[n+4] \end{aligned}$$

From Fig P2.4, the above equality is not true. Hence, the system is NOT LINEAR.

(b) To find the impulse response of the system, we note that

$$\delta[n] = x_3[n+4]$$

Therefore,

$$\begin{aligned} T\{\delta[n]\} &= y_3[n+4] \\ &= 3\delta[n+6] + 2\delta[n+5] \end{aligned}$$

(c) Since the system is known to be time-invariant and not linear, we cannot use choices such as:

$$\delta[n] = x_1[n] - x_2[n]$$

and

$$\delta[n] = \frac{1}{2}x_2[n+1]$$

to determine the impulse response. With the given information, we can only use shifted inputs.

2.36. (a) Suppose we form the impulse:

$$\delta[n] = \frac{1}{2}x_1[n] - \frac{1}{2}x_2[n] + x_3[n]$$

Since the system is linear,

$$L\{\delta[n]\} = \frac{1}{2}y_1[n] - \frac{1}{2}y_2[n] + y_3[n]$$

A shifted impulse results when:

$$\delta[n-1] = -\frac{1}{2}x_1[n] + \frac{1}{2}x_2[n]$$

The response to the shifted impulse

$$L\{\delta[n-1]\} = -\frac{1}{2}y_1[n] + \frac{1}{2}y_2[n]$$

Since,

$$L\{\delta[n]\} \neq L\{\delta[n-1]\}$$

The system is NOT TIME INVARIANT.

(b) An impulse may be formed:

$$\delta[n] = \frac{1}{2}x_1[n] - \frac{1}{2}x_2[n] + x_3[n]$$

since the system is linear,

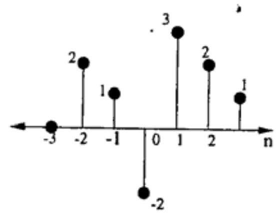
$$\begin{aligned} L\{\delta[n]\} &= \frac{1}{2}y_1[n] - \frac{1}{2}y_2[n] + y_3[n] \\ &= h[n] \end{aligned}$$

from the figure,

$$y_1[n] = -\delta[n+1] + 3\delta[n] + 3\delta[n-1] + \delta[n-3]$$
$$y_2[n] = -\delta[n+1] + \delta[n] - 3\delta[n-1] - \delta[n-3]$$
$$y_3[n] = 2\delta[n+2] + \delta[n+1] - 3\delta[n] + 2\delta[n-2]$$

Combining:

$$h[n] = 2\delta[n+2] + \delta[n+1] - 2\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$$



2.37. For an LTI system, we use the convolution equation to obtain the output:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Let $n = m + N$:

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m+N-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} x[(m-k)+N]h[k] \end{aligned}$$

Since $x[n]$ is periodic, $x[n] = x[n+rN]$ for any integer r. Hence,

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m-k]h[k] \\ &= y[m] \end{aligned}$$

So, the output must also be periodic with period N.

2.38. (a) The homogeneous solution to the second order difference equation,

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1],$$

is obtained by setting the input (forcing term) to zero.

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 0$$

Solving,

$$1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = 0,$$

$$(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})=0,$$

and the homogeneous solution takes the form

$$y_h[n]=A_1(\frac{1}{2})^n+A_2(\frac{1}{4})^n,$$

for the constants A_1 and A_2.

(b) Substituting the intial conditions,

$$y_h[-1]=A_1(\frac{1}{2})^{-1}+A_2(\frac{1}{4})^{-1}=1,$$

and

$$y_h[0]=A_1+A_2=0.$$

We have

$$2A_1+4A_2=1$$
$$A_1+A_2=0$$

Solving,

$$A_1=-1/2$$

and

$$A_2=1/2.$$

(c) Homogeneous equation:

$$y[n]-y[n-1]+\frac{1}{4}y[n-2]=0$$

Solving,

$$1-z^{-1}+\frac{1}{4}z^{-2}=0,$$

$$(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})=0,$$

and the homogeneous solution takes the form

$$y_h[n]=A_1(\frac{1}{2})^n.$$

Invoking the intial conditions, we have

$$y_h[-1]=2A_1=1$$
$$y_h[0]=A_1=0$$

Evident from the above contradiction, the initial conditions cannot be met.

(d) The homogeneous difference equation:

$$y[n]-y[n-1]+\frac{1}{4}y[n-2]=0$$

Suppose the homogeneous solution is of the form

$$y_h[n]=A_1(\frac{1}{2})^n+nB_1(\frac{1}{2})^n,$$

substituting into the difference equation:

$$y_h[n]-y_h[n-1]+\frac{1}{4}y_h[n-2]=0$$

$$A_1(\frac{1}{2})^n+nB_1(\frac{1}{2})^n-A_1(\frac{1}{2})^{n-1}-(n-1)B_1(\frac{1}{2})^{n-1}$$
$$\frac{1}{4}A_1(\frac{1}{2})^{n-2}+\frac{1}{4}(n-2)B_1(\frac{1}{2})^{n-2} = 0.$$

(e) Using the solution from part (d):

$$y_h[n] = A_1(\frac{1}{2})^n + nB_1(\frac{1}{2})^n,$$

and the initial conditions

$$y_h[-1] = 1$$

and

$$y_h[0] = 0,$$

we solve for A_1 and B_1:

$$A_1 = 0$$
$$B_1 = -1/2.$$

2.39. (a) For $x_1[n] = \delta[n]$,

$$\begin{aligned} y_1[0] &= 1 \\ y_1[1] &= ay[0] = a \end{aligned}$$

For $x_2[n] = \delta[n-1]$,

$$\begin{aligned} y_2[0] &= 1 \\ y_2[1] &= ay[0] + x_2[1] = a + 1 \neq y_1[0] \end{aligned}$$

Even though $x_2[n] = x_1[n-1]$, $y_2[n] \neq y_2[n-1]$. Hence the system is NOT TIME INVARIANT.

(b) A linear system has the property that

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}\}$$

Hence, if the input is doubled, the output must also double at each value of n. Because $y[0] = 1$, always, the system is NOT LINEAR.

(c) Let $x_3 = \alpha x_1[n] + \beta x_2[n]$.
For $n \geq 0$:

$$\begin{aligned} y_3[n] &= x_3[n] + ay_3[n-1] \\ &= \alpha x_1[n] + \beta x_2[n] + a(x_3[n-1] + y_3[n-2]) \\ &= \alpha \sum_{k=0}^{n-1} a^k x_1[n-k] + \beta \sum_{k=0}^{n-1} a^k x_2[n-k] \\ &= \alpha(h[n] * x_1[n]) + \beta(h[n] * x_2[n]) \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

For $n < 0$:

$$\begin{aligned} y_3[n] &= a^{-1}(y_3[n+1] - x_3[n]) \\ &= -\alpha \sum_{k=-1}^{n} a^k x_1[n-k] - \beta \sum_{k=-1}^{n} a^k x_2[n-k] \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

For $n = 0$:

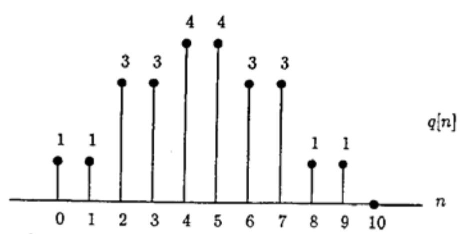
$$y_3[n] = y_1[n] = y_2[n] = 0.$$

Conclude,

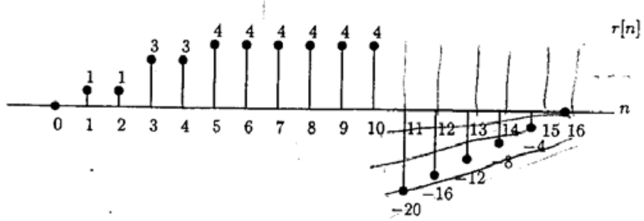
$$y_3[n] = \alpha y_1[n] + \beta y_2[n], \text{ for all } n.$$

Therefore, the system is LINEAR. The system is still NOT TIME INVARIANT.

2.50. (a) Carrying out the convolution sum, we get the following sequence $q[n]$:



(b) Again carrying out the convolution sum, we get the following sequence $r[n]$:



(c) Let $a[n] = v[-n]$ and $b[n] = w[-n]$, then:

$$
\begin{aligned}
a[n] * b[n] &= \sum_{k=-\infty}^{+\infty} a[k]b[n-k] \\
&= \sum_{k=-\infty}^{+\infty} v[-k]w[k-n] \\
&= \sum_{r=-\infty}^{+\infty} v[r]w[-n-r] \text{ where } r = -k \\
&= q[-n].
\end{aligned}
$$

We thus conclude that $q[-n] = v[-n] * w[-n]$.