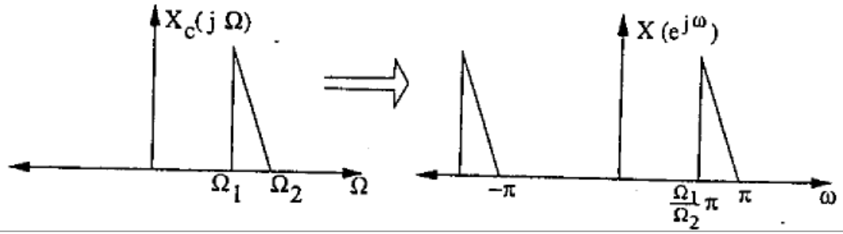
(b) We'd have to sample so that $X(e^{j\omega})$ lies between $|\omega| < \pi/2$. So $F_s \geq 4000$.

✓ **4.21.** (a) Keeping in mind that after sampling, $\omega = \Omega T$, the Fourier transform of $x[n]$ is



(b) A straight-forward application of the Nyquist criterion would lead to an incorrect conclusion that the sampling rate is at least twice the maximum frequency of $x_c(t)$, or $2\Omega_2$. However, since the spectrum is bandpass, we only need to ensure that the replications in frequency which occur as a result of sampling do not overlap with the original. (See the following figure of $X_s(j\Omega)$.) Therefore, we only need to ensure

$$\Omega_2 - \frac{2\pi}{T} < \Omega_1 \Longrightarrow T < \frac{2\pi}{\Delta\Omega}$$

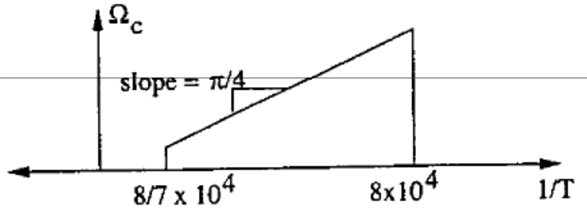
(b) Since $H_d(e^{j\omega})$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$, we don't care about any signal aliasing that occurs in the region $\frac{\pi}{4} \leq \omega \leq \pi$. We require:

$$\begin{aligned}\frac{2\pi}{T} - 2\pi \cdot 10000 &\geq \frac{\pi}{4T} \\ \frac{1}{T} &\geq \frac{8}{7} \cdot 10000 \\ T &\leq \frac{7}{8} \times 10^{-4}\text{sec}\end{aligned}$$

Also, once all of the signal lies in the range $|\omega| \leq \frac{\pi}{4}$, the filter will be ineffective, i.e., $\frac{\pi}{4} \leq T(2\pi \times 10^4)$. So, $T \geq 12.5\mu$sec.

(c)

$$\Omega = \frac{\omega}{T} \Rightarrow \Omega_c = \frac{\pi}{4T}$$



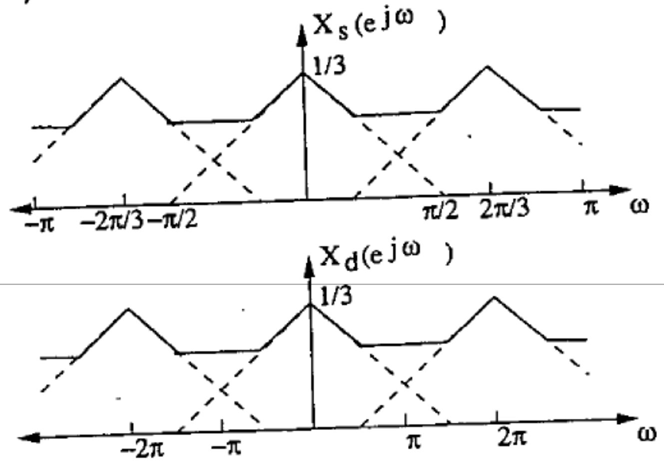
✓ **4.26.** First we show that $X_s(e^{j\omega})$ is just a sum of shifted versions of $X(e^{j\omega})$:

$$\begin{aligned}x_s[n] &= \begin{cases} x[n], & n = Mk, \quad k = 0, \pm 1, \pm 2 \\ 0, & \text{otherwise}\end{cases} \\ &= \left(\frac{1}{M}\sum_{k=0}^{M-1} e^{j(2\pi kn/M)}\right) x[n] \\ X_s(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_s[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{M}\sum_{k=0}^{M-1} x[n]e^{j(2\pi kn/M)}e^{-j\omega n} \\ &= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x[n]e^{-j[\omega-(2\pi k/M)]n} \\ &= \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j[\omega-(2\pi k/M)]}\right)\end{aligned}$$

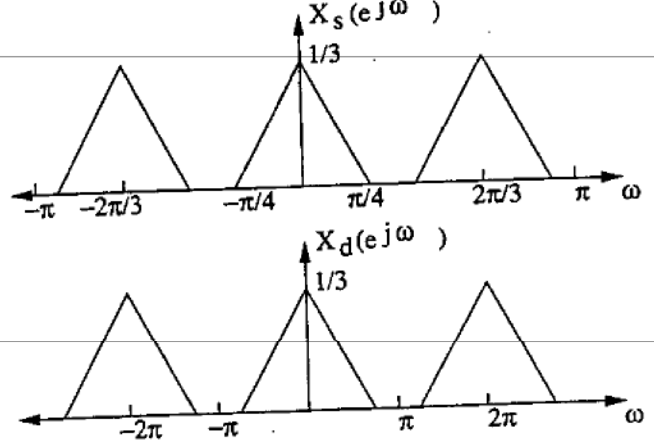
Additionally, $X_d(e^{j\omega})$ is simply $X_s(e^{j\omega})$ with the frequency axis expanded by a factor of M:

$$\begin{aligned}X_d(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} X_s[Mn]e^{-j\omega n} \\ &= \sum_{l=-\infty}^{\infty} x_s[l]e^{-j(\omega/M)l} \\ &= X_s\left(e^{j(\omega/M)}\right)\end{aligned}$$

(a) (i) $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched below for $M = 3$, $\omega_H = \pi/2$.



(ii) $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched below for $M = 3$, $\omega_H = \pi/4$.



(b) From the definition of $X_s(e^{j\omega})$, we see that there will be no aliasing if the signal is bandlimited to π/M. In this problem, $M = 3$. Thus the maximum value of ω_H is $\pi/3$.

✓**4.27.** Parseval's Theorem:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

When we upsample, the added samples are zeros, so the upsampled signal $x_u[n]$ has the same energy as the original $x[n]$:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x_u[n]|^2,$$

and by Parseval's theorem:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_u(e^{j\omega})|^2 d\omega.$$

Hence the amplitude of the Fourier transform does not change.

When we downsample, the downsampled signal $x_d[n]$ has less energy than the original $x[n]$ because some samples are discarded. Hence the amplitude of the Fourier transform will change after downsampling.

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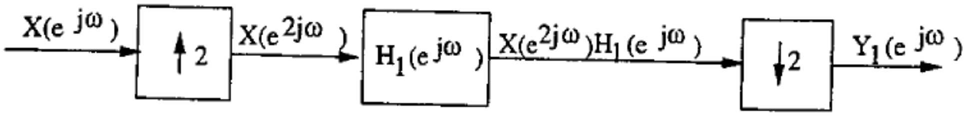
(c) No, the system is not time-invariant.

For example, suppose that $h[n] = \delta[n]$, $T = 5$ and $x_c(t) = 1$ for $-1 \leq t \leq 1$. Such a system would result in $x[n] = \delta[n]$ and $y_c(t) = \mathrm{sinc}(\pi/5)$. Now suppose we delay the input to be $x_c(t-2)$. Now $x[n] = 0$ and $y_c(t) = 0$.

√**4.29.** We can analyze the system in the frequency domain:



$Y_1(e^{j\omega})$ is $X(e^{2j\omega})H_1(e^{j\omega})$ downsampled by 2:

$$\begin{aligned}
Y_1(e^{j\omega}) &= \frac{1}{2}\left\{X(e^{2j\omega/2})H_1(e^{j\omega/2}) + X(e^{(2j(\omega-2\pi)/2})H_1(e^{j(\omega-2\pi)/2})\right\} \\
&= \frac{1}{2}\left\{X(e^{j\omega})H_1(e^{j\omega/2}) + X(e^{j(\omega-2\pi)})H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} \\
&= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} X(e^{j\omega}) \\
&= H_2(e^{j\omega})X(e^{j\omega}) \\
H_2(e^{j\omega}) &= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\}
\end{aligned}$$

4.30.

$$\begin{array}{ll}
X_c(j\Omega) = 0 & |\Omega| \geq 4000\pi \\
Y(j\Omega) = |\Omega| X_c(j\Omega), & 1000\pi \leq |\Omega| \leq 2000\pi
\end{array}$$

Since only half the frequency band of $X_c(j\Omega)$ is needed, we can alias everything past $\Omega = 2000\pi$. Hence, $T = 1/3000$ s.

Now that T is set, figure out $H(e^{j\omega})$ band edges.

$$\begin{array}{lll}
\omega_1 = \Omega_1 T & \Rightarrow \omega_1 = 2\pi \cdot 500 \cdot \frac{1}{3000} & \Rightarrow \omega_1 = \dfrac{\pi}{3} \\
\omega_2 = \Omega_2 T & \Rightarrow \omega_2 = 2\pi \cdot 1000 \cdot \frac{1}{3000} & \Rightarrow \omega_2 = \dfrac{2\pi}{3}
\end{array}$$

$$H(e^{j\omega}) = \begin{cases} |\omega| & \frac{\pi}{3} \leq |\omega| \leq \frac{2\pi}{3} \\ 0 & 0 \leq |\omega| < \frac{\pi}{3}, \frac{2\pi}{3} < |\omega| \leq \pi \end{cases}$$

4.31.

$$X_c(j\Omega) = 0, \quad |\Omega| > \frac{\pi}{T}$$

$$y_r(t) = \int_{-\infty}^{t} x_c(\tau)d\tau \Longrightarrow H_c(j\Omega) = \frac{1}{j\Omega}$$

In discrete-time, we want

$$H(e^{j\omega}) = \begin{cases} \frac{1}{j\omega}, & -\pi \leq \omega \leq \pi \\ 0, & \text{otherwise} \end{cases}$$