7.25. (a) **Answer:** Only the bilinear transform design will guarantee that a minimum phase discrete-time filter is created from a minimum phase continuous-time filter. For the following explanations remember that a discrete-time minimum phase system has all its poles and zeros inside the unit circle.

Impulse Invariance: Impulse invariance maps left-half s-plane poles to the interior of the z-plane unit circle. However, left-half s-plane zeros will *not necessarily* be mapped inside the z-plane unit circle. Consider:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} = \frac{\sum_{k=1}^{N} A_k \prod_{\substack{j=1 \\ j \neq k}}^{N} (s - s_j)}{\prod_{\ell=1}^{N} (s - s_\ell)}$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}} = \frac{\sum_{k=1}^{N} T_d A_k \prod_{\substack{j=1 \\ j \neq k}}^{N} (1 - e^{s_j T_d} z^{-1})}{\prod_{\ell=1}^{N} (1 - e^{s_\ell T_d} z^{-1})}$$

If we define $\text{Poly}_k(z) = \prod_{\substack{j=1 \\ j \neq k}}^{N} (1 - e^{s_j T_d} z^{-1})$, we can note that all the roots of $\text{Poly}_k(z)$ are inside the unit circle. Since the numerator of $H(z)$ is a sum of $A_k \text{Poly}_k(z)$ terms, we see that there are *no guarantees* that the roots of the numerator polynomial are inside the unit circle. In other words, the sum of minimum phase filters is not necessarily minimum phase. By considering the specific example of

$$H_c(s) = \frac{s + 10}{(s + 1)(s + 2)},$$

and using $T = 1$, we can show that a minimum phase filter is transformed into a non-minimum phase discrete time filter.

Bilinear Transform: The bilinear transform maps a pole or zero at $s = s_0$ to a pole or zero (respectively) at $z_0 = \frac{1 + \frac{T}{2} s_0}{1 - \frac{T}{2} s_0}$. Thus,

$$|z_0| = \left| \frac{1 + \frac{T}{2} s_0}{1 - \frac{T}{2} s_0} \right|$$

Since $H_c(s)$ is minimum phase, all the poles of $H_c(s)$ are located in the left half of the s-plane. Therefore, a pole $s_0 = \sigma + j\Omega$ must have $\sigma < 0$. Using the relation for s_0, we get

$$|z_0| = \sqrt{\frac{(1 + \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}{(1 - \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}} < 1$$

Thus, all poles and zeros will be inside the z-plane unit circle and the discrete-time filter will be minimum phase as well.

(b) **Answer:** Only the bilinear transform design will result in an allpass filter.

Impulse Invariance: In the impulse invariance design we have

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)$$

The aliasing terms can destroy the allpass nature of the continuous-time filter.

Bilinear Transform: The bilinear transform only warps the frequency axis. The magnitude response is not affected. Therefore, an allpass filter will map to an allpass filter.

(c) **Answer:** Only the bilinear transform will guarantee

$$H(e^{j\omega})|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

Impulse Invariance: Since impulse invariance may result in aliasing, we see that

$$H(e^{j0}) = H_c(j0)$$

if and only if

$$H(e^{j0}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = H_c(j0)$$

or equivalently

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0$$

which is generally not the case.

Bilinear Transform: Since, under the bilinear transformation, $\Omega = 0$ maps to $\omega = 0$,

$$H(e^{j0}) = H_c(j0)$$

for all $H_c(s)$.

(d) **Answer:** Only the bilinear transform design is guaranteed to create a bandstop filter from a bandstop filter.

If $H_c(s)$ is a bandstop filter, the bilinear transform will preserve this because it just warps the frequency axis; however aliasing (in the impulse invariance technique) can fill in the stop band.

(e) **Answer:** The property holds under the bilinear transform, but not under impulse invariance.

Impulse Invariance: Impulse invariance may result in aliasing. Since the order of aliasing and multiplication are not interchangeable, the desired identity does not hold. Consider $H_{a_1}(s) = H_{a_2}(s) = e^{-sT/2}$.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned} H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_1(z)H_2(z) \end{aligned}$$

(f) **Answer:** The property holds for both impulse invariance and the bilinear transform.

Impulse Invariance:

$$\begin{aligned} H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) \\ &= \sum_{k=-\infty}^{\infty} H_{c1}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) + \sum_{k=-\infty}^{\infty} H_{c2}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) \\ &= H_1(e^{j\omega}) + H_2(e^{j\omega}) \end{aligned}$$

Bilinear Transform:

$$\begin{aligned} H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_1(z) + H_2(z) \end{aligned}$$

(g) **Answer:** Only the bilinear transform will result in the desired relationship.
Impulse Invariance: By impulse invariance,

$$\begin{aligned} H_1\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_1}\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right) \\ H_2\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_2}\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right) \end{aligned}$$

We can clearly see that due to the aliasing, the phase relationship is not guaranteed to be maintained.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned} H_1(e^{j\omega}) &= H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right) \\ H_2(e^{j\omega}) &= H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right) \end{aligned}$$

therefore,

$$\frac{H_1(e^{j\omega})}{H_2(e^{j\omega})} = \frac{H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right)}{H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right)} = \begin{cases} e^{-j\pi/2}, & 0<\omega<\pi \\ e^{j\pi/2}, & -\pi<\omega<0 \end{cases}$$

7.26. (a) Since

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right)$$

and we desire

$$H(e^{j\omega})\,|_{\omega=0} = H_c(j\Omega)\,|_{\Omega=0},$$

we see that

$$H(e^{j\omega})|_{\omega=0} = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d}+\frac{2\pi k}{T_d}\right)\right)|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

requires

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0.$$

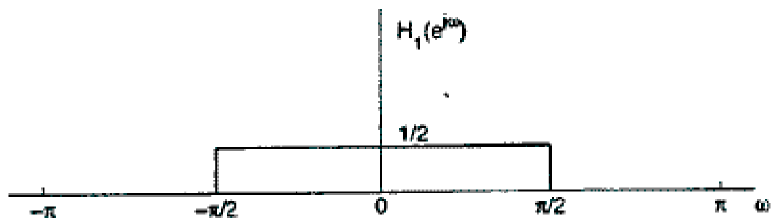
(b) Since the bilinear transform maps $\Omega=0$ to $\omega=0$, the condition will hold for any choice of $H_c(j\Omega)$.

7.27.

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega|<\frac{\pi}{4} \\ 0, & \frac{\pi}{4}<|\omega|\leq\pi \end{cases}$$

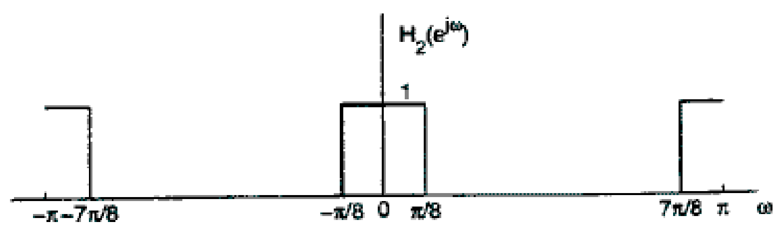
(a)

$$\begin{aligned}
h_1[n] &= h[2n] \\
H_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[2n]e^{j\omega n} \\
&= \sum_{n \text{ even}} h[n]e^{\frac{j\omega n}{2}} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[h[n] + (-1)^n h[n]\right] e^{j\frac{\omega n}{2}} \\
&= \frac{1}{2}H(e^{j\frac{\omega}{2}}) + \frac{1}{2}H\left(e^{j\frac{\omega+2\pi}{2}}\right)
\end{aligned}$$



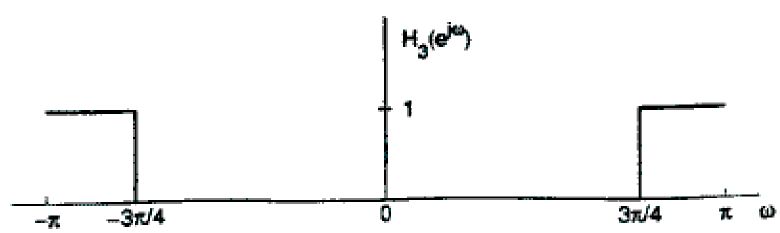
(b)

$$\begin{aligned}
H_2(e^{j\omega}) &= \sum_{n \text{ even}} h[n/2]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega 2n} \\
&= H\left(e^{j2\omega}\right)
\end{aligned}$$



(c)

$$H_3(e^{j\omega}) = H\left(e^{j(\omega+\pi)}\right)$$



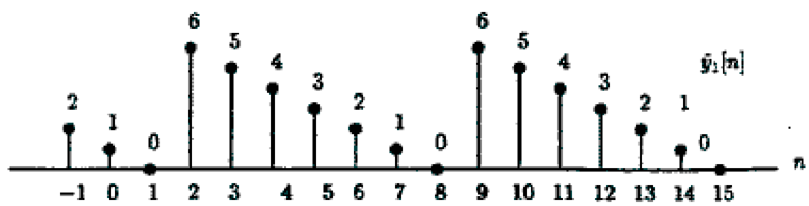
8.21. (a) We seek a sequence $\tilde{y}_1[n]$ such that

$$\tilde{Y}_1[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

From the discussion of Section 8.2.5, $\tilde{y}[n]$ is the result of the periodic convolution between $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

$$\tilde{y}_1[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

Since $\tilde{x}_2[n]$ is a periodic impulse, shifted by two, the resultant sequence will be a shifted (by two) replica of $\tilde{x}_1[n]$.



Using the analysis equation of Eq. (8.11), we may rigorously derive $\tilde{y}_1[n]$:

$$\tilde{X}_1[k] = \sum_{n=0}^{6} \tilde{x}_1[n] W_7^{kn}$$

$$
\begin{aligned}
&= 6 + 5W_7^{k} + 4W_7^{2k} + 3W_7^{3k} + 2W_7^{4k} + W_7^{5k} \\
\tilde{X}_2[k] &= \sum_{n=0}^{6} \tilde{x}_2[n] W_7^{kn} \\
&= W_7^{2k} \\
\tilde{Y}_1[k] &= \tilde{X}_1[k]\tilde{X}_2[k] \\
&= 6W_7^{2k} + 5W_7^{3k} + 4W_7^{4k} + 3W_7^{5k} + 2W_7^{6k} + W_7^{7k}
\end{aligned}
$$

Noting that $W_7^{7k} = e^{j\frac{2\pi}{7}(7k)} = 1 = W_7^{0k}$, we use the synthesis equation of Eq. (8.12) to construct $\tilde{y}_1[n]$. The result is identical to the sequence depicted above.

(b) The DFS of the signal illustrated in Fig. P8.21-2.is given by:

$$
\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{6} \tilde{x}_3[n] W_7^{kn} \\
&= 1 + W_7^{4k}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\tilde{Y}_2[k] &= \tilde{X}_1[k]\tilde{X}_3[k] \\
&= \tilde{X}_1[k] + W_7^{4k}\tilde{X}_1[k]
\end{aligned}
$$

Since the DFS is linear, the inverse DFS of $\tilde{Y}_2[k]$ is given by:

$$\tilde{y}_2[n] = \tilde{x}_1[n] + \tilde{x}_1[n-4].$$

8.22. For a finite-length sequence $x[n]$, with length equal to N, the periodic repetition of $x[-n]$ is represented by

$$x[((-n))_N] = x[((-n + \ell N))_N], \qquad \ell\text{: integer}$$

where the right side is justified since $x[n]$ (and $x[-n]$) is periodic with period N.

The above statement holds true for any choice of ℓ. Therefore, for $\ell = 1$:

$$x[((-n))_n] = x[((-n + N))_N]$$

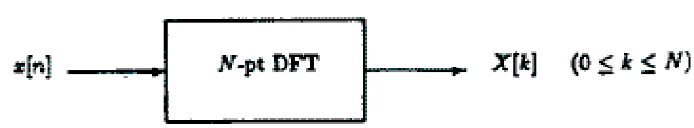
8.23. We have $x[n]$ for $0 \le n \le P$.

We desire to compute $X(z)|_{z=e^{-j(2\pi k/N)}}$ using one N-pt DFT.

(a) Suppose $N > P$ (the DFT size is larger than the data segment). The technique used in this case is often referred to as zero-padding. By appending zeros to a small data block, a larger DFT may be used. Thus the frequency spectra may be more finely sampled. It is a common misconception to believe that zero-padding enhances spectral resolution. The addition of a larger block of data to a larger DFT would enhance this quality.

So, we append $N_z = N - P$ zeros to the end of the sequence as follows:

$$
x'[n] = \begin{cases} x[n], & 0 \le n \le (P-1) \\ 0, & P \le n \le N \end{cases}
$$



(b) Suppose $N > P$, consider taking a DFT which is smaller than the data block. Of course, some aliasing is expected. Perhaps we could introduce time aliasing to offset the effects.
Consider the N-pt inverse DFT of $X[k]$,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad 0 \le n \le (N-1)$$

Suppose $X[k]$ was obtained as the result of an infinite summation of complex exponents:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{\infty} x[m]e^{-j(2\pi k/N)m}\right)W_N^{-kn}$$

Rearrange to get:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{-j(2\pi/N)(m-n)k}\right)$$

Using the orthogonality relationship of Example 8.1:

$$\begin{aligned} x[n] &= \sum_{m=-\infty}^{\infty} x[m]\sum_{r=-\infty}^{\infty}\delta[m-n+rN] \\ x[n] &= \sum_{r=-\infty}^{\infty} x[n-rN] \end{aligned}$$

So, we should alias $x[n]$ as above. Then we take the N-pt DFT to get $X[k]$.

8.24. No. Recall that the DFT merely samples the frequency spectra. Therefore, the fact the $Im\{X[k]\} = 0$ for $0 \le k \le (N-1)$ does not guarantee that the imaginary part of the continuous frequency spectra is also zero.

For example, consider a signal which consists of an impulse centered at $n = 1$.

$$x[n] = \delta[n-1], \quad 0 \le n \le 1$$

The Fourier transform is:

$$\begin{aligned} X(e^{j\omega}) &= e^{-j\omega} \\ Re\{X(e^{j\omega})\} &= \cos(\omega) \\ Im\{X(e^{j\omega})\} &= -\sin(\omega) \end{aligned}$$

Note that neither is zero for all $0 \le \omega \le 2$. Now, suppose we take the 2-pt DFT:

$$\begin{aligned} X[k] &= W_2^k, \quad 0 \le k \le 1 \\ &= \begin{cases} 1, & k = 0 \\ -1, & k = 1 \end{cases} \end{aligned}$$

So, $Im\{X[k]\} = 0, \quad \forall k$. However, $Im\{X(e^{j\omega})\} \ne 0$.

Note also that the size of the DFT plays a large role. For instance, consider taking the 3-pt DFT of

$$\begin{aligned} x[n] &= \delta[n-1], \quad 0 \le n \le 2 \\ X[k] &= W_3^k, \quad 0 \le k \le 2 \\ &= \begin{cases} 1, & k = 0 \\ e^{-j(2\pi/3)}, & k = 1 \\ e^{-j(4\pi/3)}, & k = 2 \end{cases} \end{aligned}$$

Now, $Im\{X[k]\} \neq 0$, for $k = 1$ or $k = 2$.

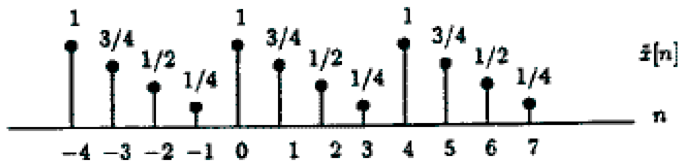
8.25. Both sequences $x[n]$ and $y[n]$ are of finite-length ($N = 4$).

Hence, no aliasing takes place. From Section 8.6.2, multiplication of the DFT of a sequence by a complex exponential corresponds to a circular shift of the time-domain sequence.

Given $Y[k] = W_4^{3k} X[k]$, we have

$$y[n] = x[((n-3))_4]$$

We use the technique suggested in problem 8.28. That is, we temporarily extend the sequence such that a periodic sequence with period 4 is formed.



Now, we shift by three (to the right), and set all values outside $0 \leq n \leq 3$ to zero.