Multiscale Analysis of Complex Time Series
by Scale Dependent Lyapunov Exponent

Jianbo Gao

Department of Electrical & Computer Engineering
University of Florida, Gainesville

gao@ece.ufl.edu
Outline

• Motivations
• Scale Dependent Lyapunov Exponent (SDLE)
  — Theory & computation
• Theoretical aspects of SDLE
  – Classification of complex motions
  – Distinguishing chaos from noise
  – Characterizing large scale orderly motions
  – Coping with nonstationarity
• Applications
• Concluding remarks
Why multiscale complexity measure?

- Complex systems are comprised of multiple subsystems that exhibit nonlinear deterministic & stochastic characteristics, and are regulated hierarchically; examples include
  - Stock markets
  - Human heart or brain
  - Weather and climate systems

- Multiscale signals with characteristics such as sensitive dependence on small disturbances, long memory, extreme variations, and nonstationarity (these terms will be defined later)

- Complex data are accumulating rapidly in health sciences, systems biology, nano-sciences, information systems, and physical sciences
How to effectively analyze complex data?

- **Highly desired**: simultaneously characterize different behaviors of the data on the entire or a wide range of scales resolvable by the data.

- Is it possible to use existing theories synergistically for this purpose?

- Extremely difficult, if not impossible
  - How to identify relevant scales to determine which theories to use?
  - How to distinguish low-dimensional chaos from noise?

- **Existing multiscale measures** ($(\varepsilon, \tau)$-entropy: Gaspard & Wang, 1993; finite size Lyapunov exponent: Aurell et al. 1997; multiscale entropy analysis: Costa et al. 2002, 2005) **cannot satisfy our need**

- Need a new readily computable measure to
  - cope with nonstationarity
  - effectively characterize structured and random aspects of the data
  - unify existing complexity measures
Sea clutter

Backscattered radar returns from a patch of sea surface
Complexities: turbulent wave motions + multipath propagation
Significance and Challenges of sea clutter modeling

- Sea clutter analysis is an important theoretical problem; may shed light on the nature of air-sea interaction

- Target detection within sea clutter is an important engineering problem (related to coastal and national security, navigation safety, and environmental monitoring) (Hu, Gao et al. 2005, 2006)


- Data are highly nonstationary, naive Fourier analysis or chaos analysis is not very useful; data are not simple random fractals
Complexities of sea clutter signals

- Signals fairly smooth on short time scales – not purely random
- Time varying frequency & randomness – highly nonstationary
- $X_t^{(m)} = (X_{tm-m+1} + \cdots + X_{tm})/m$: non-overlapping running mean of $X$ over block size $m$ — if fractal, $\text{Var}(X_t^{(m)}) \sim m^{-\alpha}$ — not the case
Heart rate variability (HRV) analysis

- HRV is an important dynamical variable of the cardiovascular function
- Most salient feature: spontaneous fluctuation, even without the presence of environmental perturbations
- HRV is related to various cardiovascular disorders
- Modeled as a fractal (1/f) process (Goldberger, Peng, Stanley, & coworkers) — to show this, data were manually segmented to remove parts that do not conform to fractal analysis
- The parameter from fractal analysis cannot perfectly distinguish healthy subjects from those with cardiac diseases
- Unknown whether cardiac dynamics are chaotic; Chaos control of fibrillation is not particularly effective
Heart rate variability (HRV) data

- Highly nonstationary — defying direct applications of existing methods
Scale dependent Lyapunov exponent (SDLE): Definition


- Consider an ensemble of trajectories in **phase space**
- Denote the initial separation between two nearby trajectories by $\varepsilon_0$, and their **average separation** at time $t$ and $t + \Delta t$ by $\varepsilon_t$ and $\varepsilon_{t+\Delta t}$, respectively
- Being defined in an average sense, $\varepsilon_t$ and $\varepsilon_{t+\Delta t}$ can be readily computed from any processes, even if they are non-differentiable
- When $\Delta t \to 0$, we have
  \[ \varepsilon_{t+\Delta t} = \varepsilon_t e^{\lambda(\varepsilon_t)\Delta t} \quad \text{or} \quad \lambda(\varepsilon_t) = \frac{\ln \varepsilon_{t+\Delta t} - \ln \varepsilon_t}{\Delta t} \]
- Given a time series data, the smallest $\Delta t = \text{sampling time}$
SDLE: Computation

- Basic idea: form proper ensemble averages
- Phase space reconstruction: $V_i = [x(i), x(i + L), \ldots, x(i + (m - 1)L)]$
  $x(i)$: given time series, $m$: embedding dimension, $L$: delay time
- Introducing a sequence of shells
  \[
  \varepsilon_k \leq \|V_i - V_j\| \leq \varepsilon_k + \Delta \varepsilon_k, \quad k = 1, 2, \ldots
  \]
  where $\varepsilon_k$ & $\Delta \varepsilon_k$ are arbitrarily chosen small distances
- 
  \[
  \lambda(\varepsilon_t) = \frac{\langle \ln \|V_{i+t+\Delta t} - V_{j+t+\Delta t}\| - \ln \|V_{i+t} - V_{j+t}\| \rangle}{\Delta t}
  \]
  Ensemble average is taken within a shell
  — add condition $|i - j| > (m - 1)L$ for chaotic systems
- Similar algorithm used in computing **time-dependent exponent curves**
Classifications of complex motions

Types of motions considered:

- Chaotic motions
  - Low-dimensional deterministic chaos
  - Noisy chaos
  - Noise-induced chaos — without noise, motion is regular

- $1/f^\beta$ processes
- $\alpha$-stable Levy processes
- Stochastic oscillations
- Complex motions with multiple scaling behaviors
Basics of chaos theory

- Chaos is also called strange attractor
- Being an attractor, trajectories in the phase space are bounded
- Being strange, nearby trajectories diverge exponentially fast: 
  \[ dr \sim dr_0 e^{\lambda_1 t} \], \( \lambda_1 \) the largest positive Lyapunov exponent
  —Sensitive dependence on initial conditions
- Kolmogorov-Sinai entropy = sum of positive Lyapunov exponent(s)
- A strange attractor typically is a fractal — non-integral dimension
- Multifractal nature of chaos: the attractor may be partitioned into many interwoven subsets, each subset has its fractal dimension, and its weight in the original attractor is well defined
SDLE $\lambda(\varepsilon)$ for chaos, noisy chaos, & noise-induced chaos

- Chaos: $\lambda(\varepsilon) \approx \text{const}$ (largest positive Lyapunov exponent)
- Noisy chaos & noise-induced chaos: $\lambda(\varepsilon) \sim -\gamma \ln \varepsilon$ on small scales
- (i) Stochastic Lorenz (63') system
  (ii) Noisy logistic map
  
  \[
  x_{n+1} = \mu x_n (1 - x_n) + P_n, \quad 0 < x_n < 1, \mu = 3.74, \sigma_{P_n} = 0.002 \\
  \text{—without noise, motion is periodic — Noise-induced chaos}
  \]
Chaotic systems with multiple positive Lyapunov exponents

- Mackey-Glass delay differential system (Science, 1977),
  \[ \frac{dx}{dt} = \frac{ax(t + \Gamma)}{1 + x(t + \Gamma)^c} - bx(t) \]

- When \( a = 0.2, b = 0.1, c = 10, \Gamma = 30 \), two positive Lyapunov exponents (the largest \( \sim 0.007 \))
$1/f^\beta$ processes

- Ubiquitous in science and engineering
  — music, turbulence, DNA sequence, human cognition, coordination, posture, visual perception, Internet traffic, device noise, etc.


- Data $\{x_t\}$: variance scales as $t^{2H}$, where $\beta = 2H + 1$

- Increment process $\{z_t = x_{t+1} - x_t\}$: power spectral density $f^{-(2H-1)}$, autocorrelation $r(k) \sim k^{2H-2}$, $k \to \infty$

- $H$: Hurst parameter
  $1/2 < H < 1$: long memory (long-range-dependence (LRD))
  $H = 1/2$: standard Brownian motion
  $0 < H < 1/2$: anti-persistence

**Basic math models for $1/f^\beta$ processes**

- Fractional Brownian motion (fBm) $B_H(t)$
  - Generalization of Brownian motion (Bm)
  - Gaussian process with mean 0 & stationary increments
  - Variance
    \[
    E[(B_H(t))^2] = t^{2H}
    \]
  - Power spectral density
    \[
    f^{-(2H+1)}
    \]
- ON/OFF intermittency
  - Power-law distributed ON/OFF train
  - $P(X \geq x) \sim x^{-\mu}$, $0 < \mu < 2$
  - $H = (3 - \mu)/2$
  - Infinite variance; also infinite mean when $0 < \mu \leq 1$
Examples of fBm processes with different $H$

(a) $H=0.25$

(b) $H=0.50$

(c) $H=0.75$

(d) $H=0.90$
Power-law scaling of $\lambda(\varepsilon)$ for $1/f^\beta$ processes

Can prove $\lambda(\varepsilon) \sim H\varepsilon^{-1/H}$

(a) fBm

(b) ON/OFF

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\lambda(\varepsilon)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 2.0$</td>
<td>slope = 3.02</td>
</tr>
<tr>
<td>$\mu = 1.6$</td>
<td>slope = 2.04</td>
</tr>
<tr>
<td>$\mu = 1.2$</td>
<td>slope = 1.45</td>
</tr>
</tbody>
</table>

| $\mu = 2.0$  | slope = 1.98            |
| $\mu = 1.6$  | slope = 1.42            |
| $\mu = 1.2$  | slope = 1.10            |
Stable laws and Levy processes

- Stable laws: distributions for the sum of independent random variables (RV’s) & those being summed have the same functional form

- Characteristic function for the distribution of a “standard” stable RV:

  \[ \Phi_z(u) = E[e^{iuZ}] = \begin{cases} 
  \exp(-|u|^\alpha[1 - j\beta \tan(\pi\alpha/2) \text{sign}(u)]), & \alpha \neq 1 \\
  \exp(-|u|^\alpha[1 + j\beta \frac{2}{\pi} \log |u| \text{sign}(u)]), & \alpha = 1,
\end{cases} \]

  where \(0 < \alpha \leq 2\): stability index; \(-1 \leq \beta \leq 1\): skewness parameter; sign: sign function (\(-1, 0, 1\) depending on whether \(u < 0, = 0, > 0\))

- In the case of strictly stable,

  \[ \sum_{i=1}^{n} Y_i \overset{d}{=} n^{1/\alpha} Y \quad \Rightarrow \quad n \text{Var}Y = n^{2/\alpha} \text{Var}Y \quad \Rightarrow \quad 0 < \alpha \leq 2 \]

- \(\alpha = 2\): normal distribution; \(0 < \alpha < 2\): heavy-tailed, \(P(X \geq x) \sim x^{-\alpha}\)

- **Generalized central limit theorem:** each stable law = attractor of the sum of independent RV’s with infinite variance
Levy processes

- Levy flights: random processes consisting of many independent steps, each step being characterized by a stable law, and consuming a unit time regardless of its length — $H = 1/2$

- Symmetric Levy flights: $\beta = 0$

- Levy walks: sampled from Levy flights with a uniform speed; each step takes time proportional to its length — the increment process similar to power-law ON/OFF train

- A symmetric $\alpha$—stable Levy flight is $1/\alpha$ self-similar

- Broad applications of Levy statistics: economics, fluid mechanics, device physics, ecology, art, etc.
Power-law scaling of $\lambda(\varepsilon)$ for Levy flights

Can prove $\lambda(\varepsilon) \sim \frac{1}{\alpha} \varepsilon^{-\alpha}$ — $\alpha$ plays the role of $1/H$
\( \lambda(\varepsilon) \) for stochastic oscillators

- Stochastic Van der Pol’s oscillator:
  \[
  \frac{dx}{dt} = y + D_1 \eta_1(t), \quad \frac{dy}{dt} = -(x^2 - 1)y - x + D_2 \eta_2(t)
  \]

- \( \lambda(\varepsilon) \sim -\ln \varepsilon \), when \((m - 1)L\) is small

- \( \lambda(\varepsilon) \sim \varepsilon^{-1/H}, \quad H \sim 1/2 \), when \((m - 1)L\) is large
Stochastic oscillations: \( H \) not necessary 1/2

- Pathological tremor: involuntary, approximately rhythmic, and roughly sinusoidal movement of parts of the body
  — often \( H > 1/2 \) (Gao & Tung, 2000; Gao 2002)

- Karman vortex street: When \( Re \approx 137, H \sim 1/2 \); When \( Re \) increases, \( H \) decreases (Lin et al. 1993; Gao 1997; Gao et al. 1999)

Left (Lim, University of Melbourne); Right: atmospheric Karman vortex street off Selkirk Island off the coast of Chile
\( \lambda(\varepsilon) \) for motions with multiple scaling behaviors

- Cencini et al. (2000): \( x_{n+1} = \lfloor x_n \rfloor + F(x_n - \lfloor x_n \rfloor) + \sigma \eta_t \)
- \( \lfloor x_n \rfloor \): integer part of \( x_n \); \( \eta_t \): uniform noise on the interval \([-1, 1]\)
  \[
  F(y) = \begin{cases} 
  (2 + \Delta)y & \text{if } y \in [0, 1/2) \\
  (2 + \Delta)y - (1 + \Delta) & \text{if } y \in (1/2, 1] 
  \end{cases}
  \]
- \( \lambda_1 = \ln(2 + \Delta) \); \( \lfloor x_n \rfloor \) introduces a random walk on integer grids — imagine a baby learning to walk
Distinguishing chaos from noise

- Popular test for chaos:
  Positive Lyapunov exponent + non-integral fractal dimension

- Chaos in brain, heart, sea clutter, ...

- Counter-example: a $1/f^{2H+1}$ process with Hurst parameter $H$ has a fractal dimension $1/H$ and a converging $K_2$ entropy

- Running on a wild beach, hunting for the **beast of chaos**
  (adapted from Broomhead & King 1988):
  “Here is a footprint”,
  “Here is another”...
  Turning around, found those are but their own footprints!

- **Key**: to classify various types of motions; to identify different scale ranges where different types of motions are manifested

- SDLE gets both done!
Characterizing large scale orderly motions by the SDLE

- Defining and characterizing large scale orderly motions, such as oscillatory ones, is a significant issue in many disciplines of science.

- Hidden frequency phenomenon:
  Fourier analysis of one variable (say $x(t)$) may not suggest any oscillatory motions, while that of another variable (say $z(t)$) may.

- How to reveal the hidden frequency? — Reconstruct a phase space using $x(t)$ to get information about $z(t)$, then get a 1-D signal.

- Existing methods
  — Ortega (1996): density method;

- Our approach: Take Fourier transform of the limiting scale, $\varepsilon_\infty$.

- Our method is more effective, because of scale-isolation.
Hidden frequency in the Lorenz (’63) system
Coping with nonstationarity

- Many real world data are nonstationary
  — HRV, economic time series, etc.
- HRV, economic data, etc., are often modeled as $1/f^{2H+1}$; however, poor scaling is usually observed when data are analyzed by standard fractal analysis methods
- Two sources of poor fractal scaling
  - “Outliers”: whose effect is to shift the $1/f^\beta$ downward or upward at randomly chosen time instances by an arbitrary amount — **type 1 nonstationarity**
  - Oscillatory events: whose effect is to concatenate or superimpose oscillatory components (on)to a $1/f^\beta$ process at randomly chosen time intervals — **type 2 nonstationarity**
- Still good scaling with SDLE
SDLE for distorted $1/f^\beta$ processes

- Type 1 nonstationarity: amounts to shifts in phase space
- Type 2 nonstationarity: affects only where $\lambda(\varepsilon) \sim 0$
Applications

- Electroencephalogram (EEG) analysis
- Analysis of heart rate variability (HRV)
- Sea clutter modeling and target detection within sea clutter
- Economic time series analysis
Analysis of EEG

- EEG signals provide a wealth of information about brain dynamics, especially related to cognitive processes and pathologies of the brain such as epileptic seizures
- Whether EEG is chaotic or not is still a big issue
- A number of complexity measures from information theory, chaos theory, and random fractal theory have been used to analyze EEG signals
- These theories have different foundations → difficult to compare among studies based on different complexity measures
- Explore the relation between SDLE & other complexity measures by analyzing windowed EEG data for the purpose of epileptic seizure prediction/detection
Complexity measures for EEG

- Information theory: Lempel-Ziv (LZ) complexity
  (major compression scheme in Unix systems; asymptotically approaches Shannon entropy; also closely related to Kolmogorov-Chaitin complexity)

- Chaos theory:
  - Lyapunov exponent (LE), computed by Wolf’s algorithm; a length scale is needed to perform normalization
  - Correlation dimension $D_2$ & correlation entropy $K_2$:
    computed by Grassberger-Procaccia’s algorithm
      \[
      C(m, \varepsilon) = \frac{1}{N^2} \sum_{i,j=1}^{N} \theta(\varepsilon - \| V_i - V_j \|) \sim \varepsilon^{D_2} e^{-mL\tau K_2}
      \]
  - Variants of $K_2$: approximate entropy, sample entropy
  - Permutation entropy (Cao, Tung, Gao, Phys. Rev. E 2004)

- Random fractal theory: Hurst parameter $H$; $H(q)$ from multifractal
Typical $\lambda(\varepsilon)$ curves for seizure and non-seizure EEG

- Human EEG data from Dr. Sackellares of Shands hospital at Univ of Florida;  \[ \lambda(\varepsilon) \sim -\ln \varepsilon \]
Complexity measures for EEG

- $\lambda_{\text{small } - \varepsilon}(t)$ & $\lambda_{\text{large } - \varepsilon}(t)$: reciprocal
- $K_2$, Approximate, sample, permutation entropy, LZ $\sim \lambda_{\text{small } - \varepsilon}(t)$
- $D_2(t) \sim \lambda_{\text{small } - \varepsilon}(t)$ —puzzling but can be resolved
- $\lambda_1(t), H(t) \sim \lambda_{\text{large } - \varepsilon}(t)$ —puzzling but can be resolved
- Scale-mixing blurs features useful for seizure prediction/detection
• PhysioNet: 18 datasets for healthy subjects, 15 datasets for subjects with congestive heart failure (CHF), a life-threatening disease

• Dimension of the dynamics of the cardiovascular system appears to be lower for healthy than for pathological subjects
  — Cardiac chaos, if does exist, is more likely to be found in healthy subjects
Distinguishing between healthy subjects and patients with CHF

- In the literature (e.g., Ivanov et al. Nature 1999; Barbieri et al. IEEE T Biomed. Eng. 2006), only 12 CHF datasets were used for this purpose
- We are able to separate all 18 healthy ones from 15 CHF ones
Sea clutter modeling & target detection within sea clutter

- ~ 400 datasets, measured with wave height 0.8 – 3.8 m, winds 0 – 90 km/hr, were obtained from Prof. Simon Haykin of McMaster Univ.

- Target: spherical block of styrofoam of size 1 m, wrapped with wire mesh.

- Small ε, \( \lambda(\epsilon) = -\gamma \ln \epsilon \); 

- Large ε, \( \lambda(\epsilon) \sim \epsilon^{-1/H} \) (not shown here, see Hu, Gao et al. 2005, 2006)
Can economic time series be modeled by low-dimensional noisy chaos?

- Controversial issue in economics since late 1980’s
  Analyzed real & simulated economic time series using the neural network based Lyapunov exponent estimator
- Found that largest Lyapunov exponent is negative
- Suggested that world economy may not be characterized by low-dimensional chaos
- Kolmogorov-Sinai entropy = sum of positive Lyapunov exponents
  **If negative, then simple regular economic dynamics**
- Economy is anything but simple!
Analysis of a chaotic asset pricing model

- Brock and Hommes asset pricing model (BH, 1998): a nonlinear map of the form:
  \[ x_t = F(x_{t-1}, x_{t-2}, x_{t-3}) + \sigma \eta_t \]
  - \( x_t \): deviation of price of the risky asset from its benchmark fundamental value; \( \sigma \eta_t \): noise

- Model contains a number of parameters

- For suitable choice of the parameter values, the model exhibits chaotic dynamics

- Hommes and Manzan (HM, 2005):
  
  Lyapunov exponent becomes negative when noise is increased

- We study two parameter sets: BH, 1998; HM, 2005
\( \lambda(\varepsilon) \) curves for the asset pricing model

- Why LE < 0 for strong stochastic forcing?
- Neural network: global optimization to minimize error between the given time series and the estimated \( \rightarrow \) involves a scale parameter \( \varepsilon^* \)
- To get non-null result, large \( \varepsilon^* \) is chosen with \( \lambda(\varepsilon^*) < 0 \)
Analysis of foreign exchange rate

- No indication of noisy chaotic behavior
- Belong to $1/f^{2H+1}$ process; majority: $H \sim 1/2$, but $H < 1/2$ or $H > 1/2$ also possible
- Deviation from $H \sim 1/2$ might indicate economic/political ties between two nations

![Diagram](image)

\[ H = 0.49 \pm 0.04 \]
\[ H = 0.43 \pm 0.10 \]
\[ H = 0.66 \pm 0.05 \]
Concluding remarks

- Introduced a new multiscale complexity measure, the SDLE
- Considered issues of classification of complex motions, distinction between chaos & noise, nonstationarity, hidden frequency
- Considered various applications — results are all promising
- What is next?
  - Meaning of $\gamma$ in $\lambda(\varepsilon) \sim -\gamma \ln \varepsilon$? — **Rate of loss of information**
  - Spatial-temporal systems
  - Information flow & directional control
    (neuroscience — cortical control of advanced prostheses; systems biology — cell cycle control, cell-cell communication)
- **Contact me if you are interested in our methods and codes!**
Joint work with

**Jing Hu**: University of Florida

**Wen-wen Tung**: Purdue University

**Yinhe Cao**: Biosieve

Partially supported by DOE & ARO

**Jing Hu** will graduate soon. She is now looking for a good job.