# Mandelbrot Set



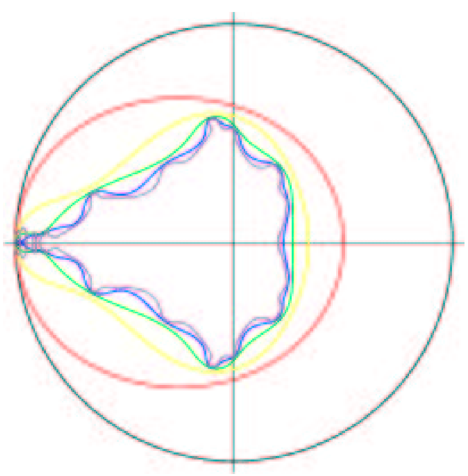
The term Mandelbrot set is used to refer both to a general class of fractal sets and to a particular instance of such a set. In general, a Mandelbrot set marks the set of points in the complex plane such that the corresponding Julia set is connected and not computable.

"The" Mandelbrot set is the set obtained from the quadratic recurrence equation

$$z_{n+1} = z_n^2 + C \qquad (1)$$

with $z_0 = C$, where points C in the complex plane for which the orbit of $z_n$ does not tend to infinity are in the set. Setting $z_0$ equal to any point in the set that is not a periodic point gives the same result.

The Mandelbrot set was originally called a $\mu$ molecule by Mandelbrot. J. Hubbard and A. Douady proved that the Mandelbrot set is connected. Shishikura (1994) proved that the boundary of the Mandelbrot set is a fractal with Hausdorff dimension 2. However, it is not yet known if the Mandelbrot set is pathwise-connected. If it is pathwise-connected, then Hubbard and Douady's proof implies that the Mandelbrot set is the image of a circle and can be constructed from a disk by collapsing certain arcs in the interior (Douady 1986). The area of the set is known to lie between 1.5031 and 1.5702; it is estimated as 1.50659....



To visualize the Mandelbrot set, the limit at which points are assumed to have escaped can be approximated by $r_{\max}$ instead of infinity. Beautiful computer-generated plots can be then be created by coloring nonmember points depending on how quickly they diverge to $r_{\max}$. A common choice is to define an integer called the count to be the largest n such that $|z_n| < r_{\max}$, where r can be conveniently taken as $r_{\max} = 2$, and to color points of different count different colors. The boundary between successive counts defines a series of "Mandelbrot set lemniscates" (or "equipotential curves"; Peitgen and Saupe 1988) defined by iterating the quadratic recurrence,

$$L_n(C) = z_n = r_{\max}. \quad (2)$$

The first few lemniscates are therefore given by

$$L_1(C) = C \quad (3)$$

$$L_2(C) = C(C+1) \quad (4)$$

$$L_3(C) = C + (C + C^2)^2 \quad (5)$$

$$L_4(C) = C + [C + (C + C^2)^2]^2. \quad (6)$$

When writing $C = x + iy$ and taking the absolute square of each side, the lemniscates can plotted in the complex plane, and the first few are given by

$$r^2 = x^2 + y^2 \quad (7)$$

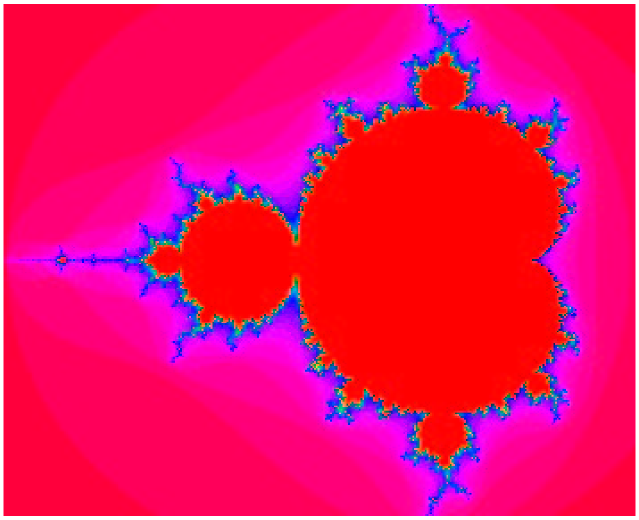
$$r^2 = (x^2 + y^2)[(x+1)^2 + y^2] \quad (8)$$

$$r^2 = (x^2 + y^2)(1 + 2x + 5x^2 + 6x^3 + 6x^4 + 4x^5 + x^6 - 3y^2 - 2xy^2 + 8x^2y^2 + 8x^3y^2 + 3x^4y^2 + 2y^4 + 4xy^4 + 3x^2y^4 + y^6). \quad (9)$$

These are a circle (black), an oval (red), and a pear curve (yellow). In fact, the second Mandelbrot set lemniscate $L_2$ can be written in terms of a new coordinate system with $x' \equiv x - 1/2$ as

$$[(x' - \tfrac{1}{2})^2 + y^2][(x' + \tfrac{1}{2})^2 + y^2] = r^2, \quad (10)$$

which is just a Cassini oval with $a = 1/2$ and $b^2 = r$. The Mandelbrot set lemniscates grow increasingly convoluted with higher count, illustrated above, and approach the Mandelbrot set as the count tends to infinity.

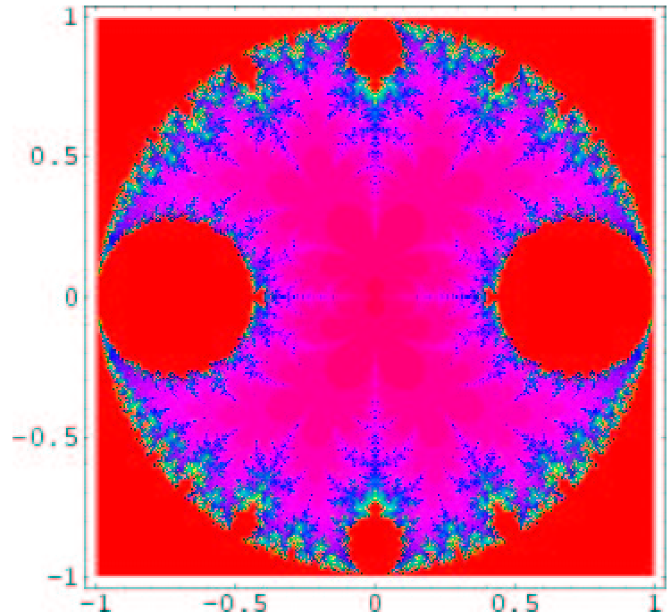


A plot of the Mandelbrot set is shown above in which values of C in the complex plane are colored according to the number of steps required to reach $r_{\max} = 2$. The kidney bean-shaped portion of the Mandelbrot set turns out to be bordered by a cardioid with equations

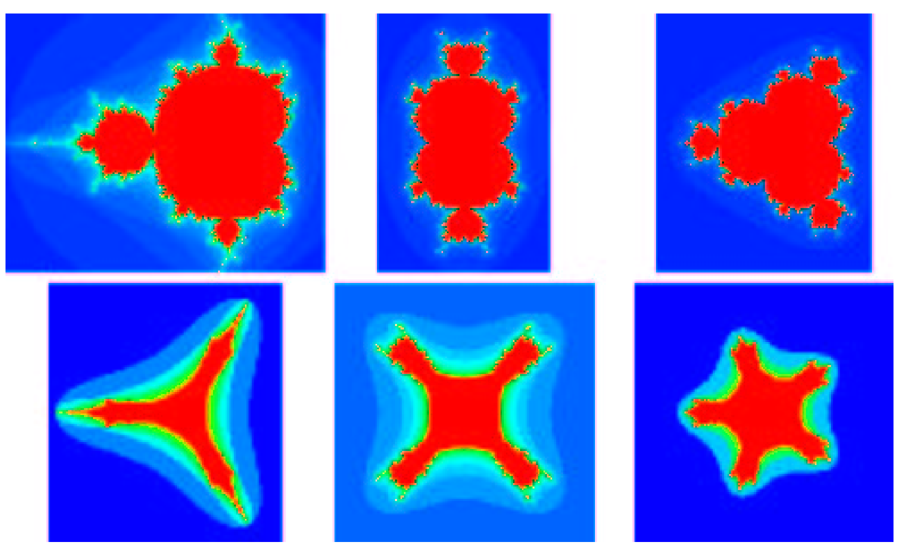
$$4x = 2\cos t - \cos(2t) \quad (11)$$

$$4y = 2\sin t - \sin(2t). \quad (12)$$

The adjoining portion is a circle with center at $(-1, 0)$ and radius $1/4$. One region of the Mandelbrot set containing spiral shapes is known as sea horse valley because the shape resembles the tail of a sea horse.




The term Mandelbrot set can also be applied to generalizations of "the" Mandelbrot set in which the function $f(z) = z^2 + C$ is replaced by some other function. In the above plot, $f(z) = \sin(z/c)$, $z_0 = c$, and c is allowed to vary in the complex plane. Note that completely different sets (that are not Mandelbrot sets) can be obtained for choices of $z_0$ that do not lie in the fractal attractor. So, for example, in the above set, picking $z_0$ inside the unit disk but outside the red basins gives a set of completely different-looking images.



Generalizations of the Mandelbrot set can be constructed by replacing $z_n^2$ with $z_n^k$ or $(\bar{z}_n)^k$, where k is a positive integer and $\bar{z}$ denotes the complex conjugate of z. The above figures show the fractals obtained for k = 2, 3, and 4 (Dickau). The plots on the bottom have z replaced with $\bar{z}$ and are sometimes called "mandelbar sets."

**SEE ALSO:** Cactus Fractal, Fractal, Julia Set, Mandelbar Set, Mandelbrot Set Lemniscate, Quadratic Map, Randelbrot Set, Sea Horse Valley

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## CITE THIS AS:

Eric W. Weisstein. "Mandelbrot Set." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/MandelbrotSet.html