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Direct Dynamical Test for Deterministic Chaos.

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Abstract. – We present a direct and dynamical method to distinguish low-dimensional deterministic chaos from noise. We define a series of time-dependent curves which are closely related to the largest Lyapunov exponent. For a chaotic time series, there exists an envelope to the time-dependent curves, while for a white noise or a noise with the same power spectrum as that of a chaotic time series, the envelope cannot be defined. When a noise is added to a chaotic time series, the envelope is eventually destroyed with the increasing of the amplitude of the noise.

Complex time series are ubiquitous in nature and in man-made systems, and a variety of measures have been proposed to characterize them. Among the most widely used approaches today are state space reconstruction by the time delay embedding [1], calculation of the correlation dimension and of the K_2 entropy [2, 3], and estimation of the Lyapunov exponents [4, 5], for characterizing strange attractors. Since low-dimensional strange attractors produce a small and usually non-integer value of the dimension and a converging entropy, and a positive largest Lyapunov exponent, in practice these have often been taken as «proof» of the presence of a strange attractor. However, there exist some stochastic processes which generate time series with finite correlation dimension and converging K_2 entropy estimates [6-8]. Hence, it is very important to develop a method for distinguishing noise from chaos in an observed time series and gain an insight into the system under investigation.

There exist several statistical ways of detecting deterministic processes. Kaplan and Glass [9] have used a coarse-grained directional vector, and Wayland *et al.* have employed «phase space continuity» [10], which is a variant of the Kaplan-Glass method, while Sugihara and May [11] and Kennel and Isebelle [12] have explored prediction to detect determination in a time series. In this paper we shall extend the concept of the time-dependent exponent [13] and develop a direct and dynamical test for distinguishing chaos from noise.

Having a scalar time series $\{x_i\}$, $i = 1, 2, \ldots$, with sampling time δt, one can construct vectors of this form: $X_i = (x_i, x_{i+L}, \ldots, x_{i+(m-1)L})$, with m the embedding dimension and L

the delay time [1]. Assume a proper reconstruction of the state space has been achieved, *i.e.* m is greater than or equal to the minimal acceptable embedding dimension and L is also chosen properly [13-16], then a dynamics F: $X_i \to X_{i+1}$ is constructed. The time-dependent exponent $\Lambda(k)$ is defined by [13]

$$\Lambda(k) = \langle \ln (\|X_{i+k} - X_{j+k}\| / \|X_i - X_j\|) \rangle, \qquad \|X_i - X_j\| \leq r^* . \tag{1}$$

The angle brackets denote ensemble average of all possible pairs of (X_i, X_j), r^* is a prescribed sufficiently small distance, and $k\,\delta t$ is the evolution time.

Λ is a function of k, m, and L. For fixed small k, by requiring that F be a continuous mapping preserving neighbourhood relations, the minimal acceptable embedding dimension can be determined by requiring that $\Lambda(m)$ does not decrease significantly when further increasing m, and an optimal L is obtained by the minimum of $\Lambda(L)$ which implies that the orbital motion is uniform and the distortion is small [13]. This is the so-called optimal embedding. When this has been achieved, for small values of k, $\Lambda(k)/k\,\delta t$ measures the mixture of all local Lyapunov exponents [17]. However, the largest Lyapunov exponent eventually dominates the dynamics with increasing k; in other words, the small separation vector between X_i and X_j will align with the eigendirection for the largest Lyapunov exponent. Actually, $\Lambda(k)/k\,\delta t$ gives the standard estimation of this exponent for large k, which is a consequence of the ergodicity of the motion on the attractor [13]. Geometrically, this is to say that the $\Lambda(k)$ curve is a straight line for $k\,\delta t$ large and nearly passes through the origin when extrapolated, the slope of the linear $\Lambda(k)$ curve estimates the largest Lyapunov exponent.

As can be imagined easily, a quantity like $\Lambda(k)/k\,\delta t$ calculated from noise may also be positive. However, for a stochastic process $\Lambda(k)$ cannot be expected to be linear in k for k not small. Actually for an IID (independent with identical distribution) random variable series, we have the following functional forms ([1]):

$$\Lambda = \begin{cases} f(k, m, r^*), & 1 \leq k \leq m-1, \\ g(m, r^*), & k \geq m. \end{cases} \tag{2}$$

Hence we see that $\Lambda(k)$ as a whole cannot be linear in k for a white noise. Rather, Λ depends on r^* and m. We also note that $\Lambda(k)$ is always positive if r^* is very small, because $\|X_{i+k} - X_{j+k}\|$ has greater probability to be larger than $\|X_i - X_j\|$. By the same reasoning we also know that $\Lambda(k)$ increases with increasing k when $1 \leq k \leq m-1$. When $k \geq m$, however, we conclude that the $\Lambda(k)$ curve is a horizontal line when there are many pairs of (X_i, X_j) to well define the ensemble average.

An implication is that if we calculate the largest Lyapunov exponent from a white noise, we would always have a positive number. However, this number does not imply chaos, since it depends on r^* and m, and probably also on L for a coloured noise. Actually this number can become as large as one desires provided that the data size is so large that r^* can be very small.

To gain an insight into the dependence of the $\Lambda(k)$ curve on r^* for a noise and to deal with noise-contaminated data with unknown noise level, we extend formulation (1) by defining $\Lambda(k)$ on a series of shells, $r_{i+1} \leq \|X_i - X_j\| \leq r_i$, and calculate the corresponding $\Lambda(k)$ curves. We take the Lorenz equations ($\dot{x} = \sigma(y - x)$, $\dot{y} = x(r - z) - y$, $\dot{z} = xy - bz$, $\sigma = 10$, $b = 8/3$, $r = 45.92$, $\delta t = 0.06$) and the Mackey-Glass equation ($\dot{x} = ax(t + \Gamma)/[1 + x(t + \Gamma)^c] - bx(t)$,

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([1]) Proof will be given in a longer report.

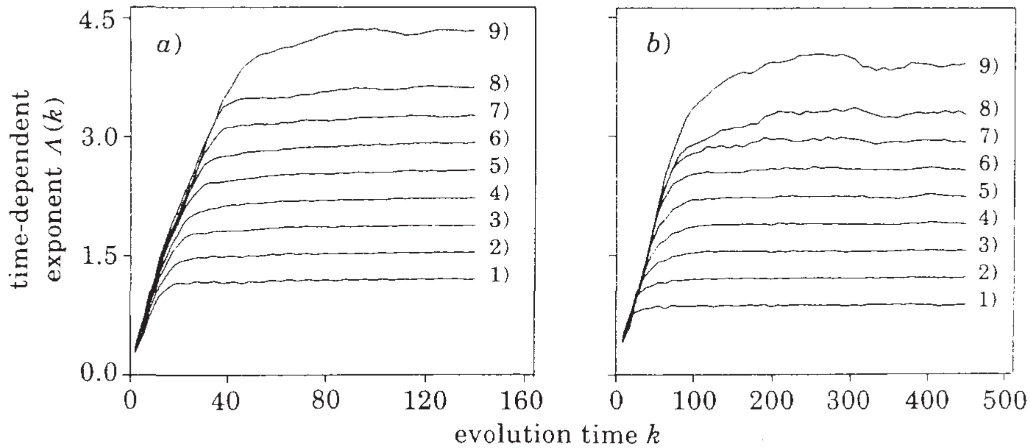

Fig. 1. – The $\Lambda(k)$ curves for *a*) the Lorenz system ($m = 4$, $L = 2$) and *b*) the Mackey-Glass equation ($m = 5$, $L = 1$). The time series is normalized to (0, 1). 5000 data points are used. Curve 1) to 9) correspond to shells $(2^{-i-1}, 2^{-i})$, with *a*) Lorenz system, $i = 5, 6, \ldots, 13$, and *b*) Mackey-Glass equation, $i = 3, 4, \ldots, 11$, respectively.

$a = 0.2$, $b = 0.1$, $c = 10$, $\Gamma = 30$, $\delta t = 6$), which have one and two positive Lyapunov exponents, respectively, as two examples to illustrate some results which are typical for other model chaotic systems. We notice from fig. 1 that there exists a linear envelope to the $\Lambda(k)$ curves. The slope of the envelope estimates the largest Lyapunov exponent of the two systems to be 1.50 and 0.007, which are quite standard [13]. As a matter of fact, the linear segment of the $\Lambda(k)$ curve for the smallest shell (which is a ball) is just the $\Lambda(k)$ curve calculated from expression (1), and the existence of the envelope reflects the fractal nature of the attractor. Also we see that a time scale of dynamical correlation (corresponding to the linear increasing segment of the $\Lambda(k)$ curves) is associated with each of the shells, which is important for prediction. Beyond that time scale, the chaotic motion is indistinguishable from a stochastic process.

We now turn to discuss stochastic processes. Figure 2 gives a typical result for a white noise with uniform distribution, which shows all the features pointed out through expression (2). Figure 3 shows the result for the surrogate data of the Lorenz system, *i.e.* a data set with the same spectrum but randomized Fourier phases. The qualitative features of fig. 3 are

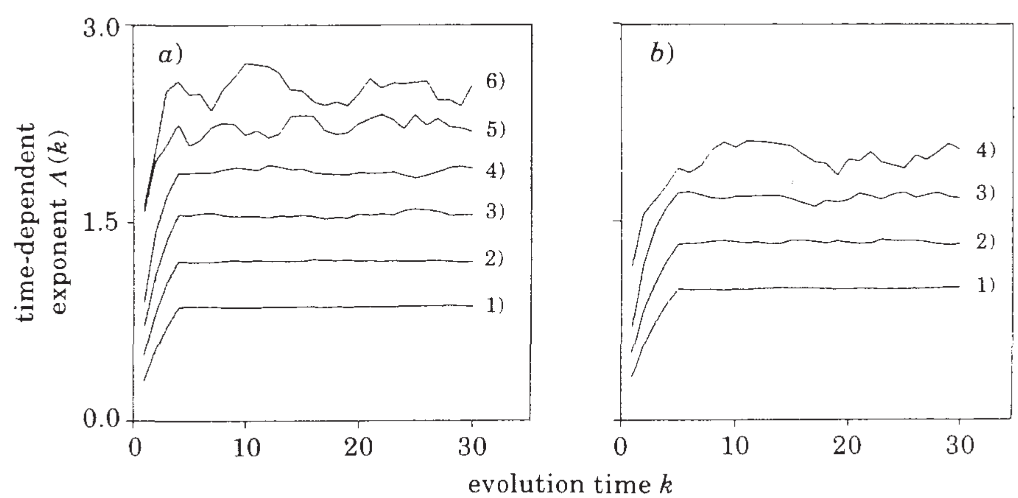

Fig. 2. – The $\Lambda(k)$ curves for a normalized uniformly distributed white noise (*a*) $m = 4$, *b*) $m = 5$). 6000 data points are used. Curves 1) to 9) correspond to shells $(2^{-i-1}, 2^{-i})$, $i = 4, 5, \ldots, 12$.

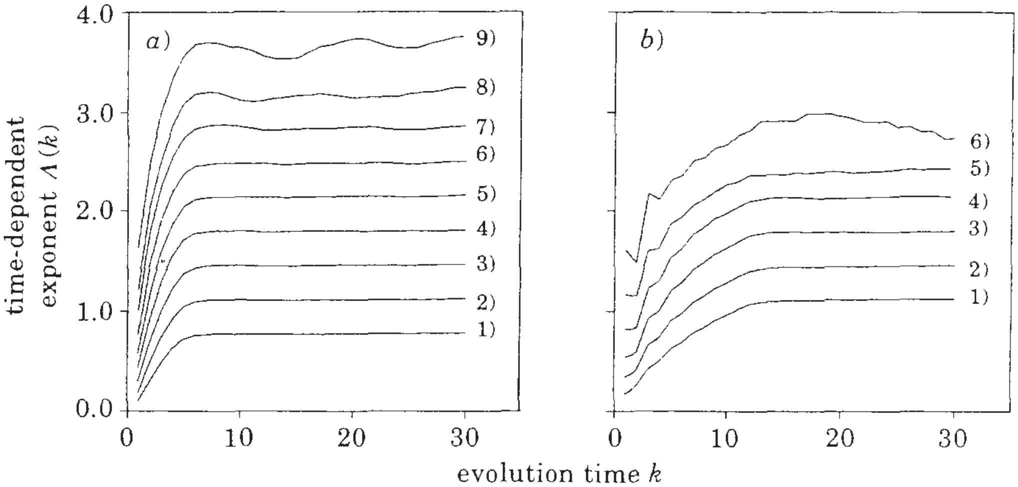

Fig. 3. – The $\Lambda(k)$ curves for the surrogate data of the Lorenz system (a) $m = 4$, $L = 1$; b) $m = 6$, $L = 2$). The time series is normalized to $(0, 1)$ and 6000 data points are used. Curves 1) to 9) correspond to shells $(2^{-i-1}, 2^{-i})$, $i = 4, 5, \ldots, 12$.

similar to fig. 2, with the new characteristics that $\Lambda(k)$ depends on L and a time scale corresponding to the increasing of $\Lambda(k)$ slightly larger than the embedding window $(m - 1)L$ which is due to the conditional probability caused by the embedding procedure and the colourness of the noise. The most important fact is that due to the dependence of the $\Lambda(k)$ curves on the radii of the shells, an envelope to the $\Lambda(k)$ curves no longer exists, and the largest positive Lyapunov exponent cannot be defined. Another important fact is that if the time scale corresponding to the increasing of $\Lambda(k)$ is taken as the prediction time, then it is significantly smaller than the time scale of dynamical correlation given by fig. 1.

A note on the value of m used to obtain the results of fig. 1 needs to be made. Both for the Lorenz system and the Mackey-Glass equation, m is chosen to be the minimal acceptable embedding dimension plus one. The results of fig. 1 do not change when m is further increased. However, if m is smaller than the minimal acceptable embedding dimension, then the $\Lambda(k)$ curves behave more or less like that of a noise, and the envelope no longer exists.

Having distinguished clean chaotic signal from pure noise, we now discuss time series of this type, $\{x_i\} + a\{\eta_i\}$, $\{x_i\}$ is a clean chaotic signal and $\{\eta_i\}$ is a pure noise, both of which are normalized to $(0, 1)$, and a is the noise level. Intuitively one can imagine that the $\Lambda(k)$ curves for shells of smaller radii will take the characteristics of fig. 2 and 3, while for shells of larger radii, the characteristics of fig. 1 will be preserved, *i.e.*, there will only exists a kind of envelope to $\Lambda(k)$ curves of larger-radii shells. The higher the noise level, the more the envelope is destroyed. When the noise level is too high, it may be difficult to extract the characteristic of the chaotic motion, since the largest acceptable radius of the shell is bounded by the upper bound $r_{\max}$ meaningful for the calculation of the fractal dimension, as pointed out by Eckmann and Ruelle [5]. Figure 4 shows typical results for the Lorenz system supplemented with its surrogate data, which confirm the qualitative features described above. We also note that when $a \geq 0.2$, the characteristics of the chaotic motion are already very difficult to identify. The characteristics of fig. 4 remain similar when a white noise of Gaussian distribution or uniform distribution is added.

In conclusion, chaotic motion is characterized by a linear $\Lambda(k)$ curve, the slope of the curve yields an estimate of the largest Lyapunov exponent. Hence, there exists an envelope, which is linear or nearly linear, to the $\Lambda(k)$ curves defined on a series of shells. For stochastic

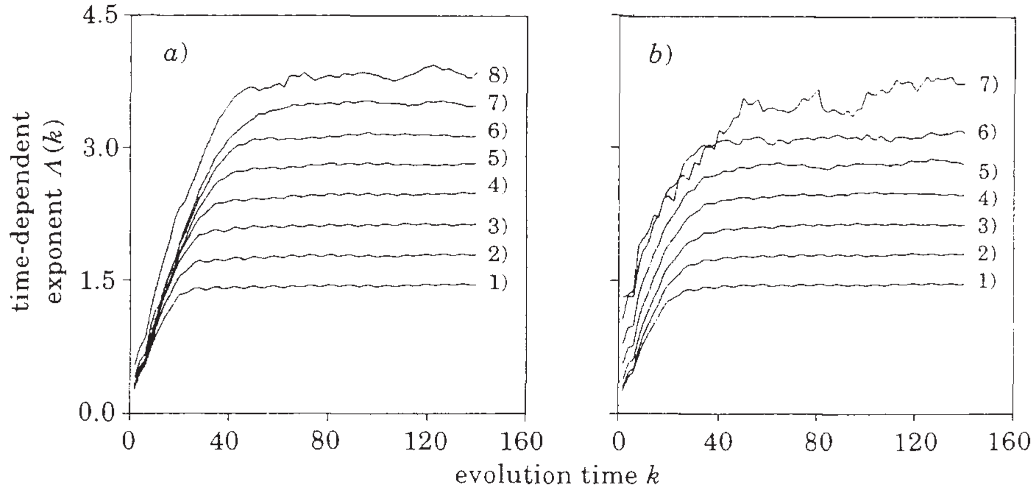

Fig. 4. – The $\Lambda(k)$ curves for the Lorenz system added with its surrogate data (a) $m = 6$, $a = 0.5$; b) $m = 6$, $a = 0.2$). The time series is normalized to (0, 1) and 6000 data points are used. Curves 1) to 8) correspond to shells $(2^{-i-1}, 2^{-i})$, $i = 5, 6, \ldots, 12$, respectively.

processes, $\Lambda(k)$ cannot be linear in k, and the value of Λ depends on the radii of the shells. Therefore, there no longer exists an envelope to the $\Lambda(k)$ curves. This clear difference provides a direct and dynamical method of distinguishing chaos from stochastic processes. When a noise is added to a chaotic signal, the envelope to the $\Lambda(k)$ curves of smaller radii for the underlying chaotic system is destroyed. The higher the noise level, the more the envelope is destroyed.

As a final remark, we point out that when the noise level is high up to 20%, the envelope to the $\Lambda(k)$ curves is destroyed (fig. 4). However, the time scale corresponding to the dynamical one of fig. 1 is nearly preserved. We expect that an appropriate statistic which incorporates the significant difference between the dynamical time scale of fig. 1 and the embedding time scale of fig. 2 and 3 can be developed to deal with noise-contaminated data with much higher noise level.

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