When Can Noise Induce Chaos?

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Crutchfield *et al.* observed that noise truncates period-doubling cascade and induces chaos. To date, however, very little low-dimensional chaos has been unambiguously identified experimentally. This discrepancy stimulates us to reexamine the noisy logistic map. We find that noise can indeed induce chaos. However, this is not associated with the main 2^n cascade. We identify three basic conditions for noise to induce chaos. We also show that when noise induces chaos the complete period-doubling cascade is inhibited, otherwise the cascade is simply masked by noise. [S0031-9007(99)08394-5]

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Noise can induce a number of interesting phenomena in nonlinear dynamical systems, such as noise-induced order [1] and noise-induced chaos [2–4]. The latter was first observed in a driven nonlinear oscillator [3], and later studied via the noisy logistic map [4]. The main idea is that intrinsic noise truncates the period-doubling cascade. That is, the periodic motions with high periods of the clean system are replaced by chaoslike motions when there is noise. Indeed, when a period-doubling cascade is observed in experimental situations, such as in fluid flows [5], semiconductor lasers [6], chemical reactions [7], and biological systems [8], and in a plasma reactor [9], only the first few period-doubling bifurcations can be observed.

Since the period-doubling cascade is a universal feature of nonlinear dynamical systems, we would expect that chaos associated with this cascade should be readily observed experimentally. Because of the difficulty in distinguishing between low-dimensional chaos and noise [10], however, to date, not many true low-dimensional chaotic systems have been identified experimentally. This discrepancy leads us to ask a series of questions: Can an experimentally observed chaoslike motion associated with a period-doubling cascade be unambiguously identified as deterministic and low dimensional? Can noise indeed induce chaos? If noise can induce chaos, then when can this happen? Since noise truncates the period-doubling cascade, is the complete period-doubling cascade inhibited, or just masked by noise? To answer these questions, we study the following noisy logistic map:

$$x_{n+1} = \mu x_n(1 - x_n) + P_n, \qquad 0 < x_n < 1, \qquad (1)$$

where μ is the bifurcation parameter and P_n is a Gaussian random variable with zero mean and standard deviation σ. We will refer to σ as the noise level. The bifurcation diagrams, both for the clean and noisy systems, can be found in Crutchfield *et al.* [4]. Here, we will study the system behavior at parameter values $\mu = 3.55$, 3.63, 3.74, and 3.83. The clean system at these parameter values is periodic with periods 8, 6, 5, and 3, respectively. Note that $\mu = 3.55$ belongs to the main 2^n cascade, while $\mu = 3.83$ belongs to the period(3)-doubling cascade.

To determine whether noise can induce chaos or not, we need to define chaos carefully. Mathematically speaking, a noisy system, no matter how small the noise is, has infinite dimensions. Experimentally speaking, one would be more interested in a certain range of finite scales. If the noise is very weak, then its influence on the dynamics may be limited to very small scales, leaving the dynamics on finite scales deterministiclike. Here we will adopt this experimentalist's point of view, and define chaos by the exponential divergence between nearby trajectories on certain finite scales.

Let us be more quantitative. From the time series $\{x(i)\}$ of the noisy logistic map, we first construct vectors $\{X_i\}$ by the time delay embedding technique [11]: $X_i = [x(i), x(i+L), \ldots, x(i+(m-1)L)]$, with m being the embedding dimension and L being the delay time (we choose m to be 4 and L to be 1 here). We then compute the time-dependent exponent $\Lambda(k)$ curves [12],

$$\Lambda(k) = \left\langle \ln\left(\frac{\|X_{i+k} - X_{j+k}\|}{\|X_i - X_j\|} \right) \right\rangle, \qquad (2)$$

with $r \le \|X_i - X_j\| \le r + \Delta r$, where r and Δr are prescribed small distances. The angle brackets denote ensemble averages of all possible pairs of (X_i, X_j). The integer k, called the evolution time, corresponds to time $k\delta t$. Note that geometrically $(r, r + \Delta r)$ defines a shell, and a shell captures the notion of scale. For clean chaotic systems, the $\Lambda(k)$ curves first increase linearly with k till some predictable time scale, k_p, then flattens [13]. The linearly increasing parts of the $\Lambda(k)$ curves corresponding to different shells collapse together to form an envelope. This property forms a direct dynamical test for deterministic chaos [12]. For noisy chaotic systems, the linearly increasing part of the $\Lambda(k)$ curves corresponding to small shells break themselves away from the envelope. The stronger the noise, the more $\Lambda(k)$ curves break away from the envelope. Only if the noise is not too strong so that the linearly increasing parts of the $\Lambda(k)$ curves corresponding to some finite scale shells still collapse together, can we say that the motion is

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chaotic. Noise-induced chaos should also be defined in this manner.

Crutchfield *et al.* [4] suggested that the effect of noise is to average the structure of deterministic attractors over some range of nearby parameters. For noise to be able to induce a transition from a certain periodic state (corresponding to some parameter μ_0) to a chaotic state, the chaotic motions corresponding to adjacent parameter values should still behave chaotically in the presence of such noise. If those adjacent chaotic motions are very insensitive to noise, then we can anticipate that noise-induced chaos is likely to occur. This argument suggests that, to answer whether noise can induce chaos or not, it would first be useful to clearly understand how noise affects chaos.

For a quantitative understanding of how noise affects chaos, it is more convenient to work with the logarithmic displacement curves instead of the time-dependent exponent $\Lambda(k)$ curves [13]. This is done by rewriting Eq. (2) as

$$D(k) = \langle \ln \|X_{i+k} - X_{j+k}\| \rangle = \langle \ln \|X_i - X_j\| \rangle + \Lambda(k), \quad (3)$$

and plotting $\langle \ln \|X_{i+k} - X_{j+k}\| \rangle$ as a function of the evolution time k. Now the linearly increasing parts of the curves corresponding to different shells separate, representing the short-term memory of the chaotic system. By adding noise, the separation shrinks, reflecting loss of memory. The stronger the noise, the more the separation shrinks. This can be quantified by taking the ratio of the separation between the displacement curves for the noisy and clean systems. More concretely, we take two logarithmic displacement curves corresponding to different shells, denote them as D_1 and D_2, calculate the area between them, do this for both the noisy and the clean systems, and take their ratio. Since this ratio is a normalized area, we denote it by NA. This procedure can be well approximated by the following formula [13]:

$$\mathrm{NA} \approx \frac{\sum_i [D_1(k_i) - D_2(k_i)]|_{\text{with-noise}}}{\sum_i [D_1(k_i) - D_2(k_i)]|_{\text{without-noise}}}, \quad (4)$$

with $k_i > (m-1)L$, $i = 1, 2, 3, \ldots$. NA typically decreases from 1 to 0 with the strength of the noise. Here we are more interested in how some noise of fixed strength affects different chaotic states. This can be readily done by computing NA for different parameters μ for certain fixed noise levels. Figure 1 shows, for noise levels $\sigma = 0.002$ (open triangles) and 0.003 (filled circles), the variation of NA with μ. Note that at about $\mu = 3.63$, 3.74, and 3.83, the clean logistic map has periodic windows. They are reflected as dips in the NA vs μ curves at about $\mu = 3.63$, 3.74, and 3.83. Also note that NA for some chaotic states is larger than 1. This is caused by the fact that with noise the slope of the linearly increasing parts of the $\Lambda(k)$ curves corresponding to those parameter values increases, resulting in an enlarged positive Lyapunov exponent. This phenomenon may be termed "noise-enhanced chaos." While this was a general conclusion by Crutchfield *et al.* [4], we see here that this is true only for certain chaotic states. Nevertheless, this fact upholds our hope that noise-induced chaos is likely to happen.

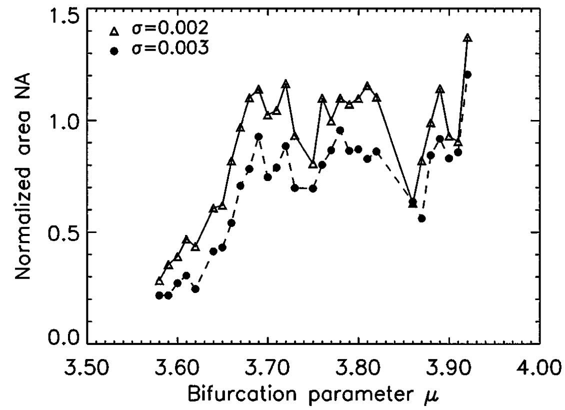

FIG. 1. Variation of the normalized area NA with the bifurcation parameter μ for two noise levels $\sigma = 0.002$ (open triangles) and 0.003 (filled circles). Shells $(2^{-(i+1)/2}, 2^{-i/2})$ with $i = 7$ and 8 were used in the computation.

Recall that we have speculated that for noise to induce chaos the chaotic states adjacent to certain periodic states have to be insensitive to noise. Thus, based on Fig. 1, we anticipate that noise-induced chaos is likely to occur at about $\mu = 3.74$, but unlikely to happen at about $\mu = 3.57$, which belongs to the main period(2)-doubling cascade. This idea can be readily tested by computing the $\Lambda(k)$ curves for the noisy logistic map at $\mu = 3.55$, 3.63, 3.74, and 3.83. For each case, we can adjust the noise level till the $\Lambda(k)$ curves best show chaoslike features. The results are shown in Figs. 2(a)–2(d), where, for each figure, six curves, from bottom to top, correspond to shells $(r, r + \Delta r)_i = (2^{-(i+1)/2}, 2^{-i/2})$ with $i = 7, 8, \ldots, 12$. Clearly, the linearly increasing segments of the $\Lambda(k)$ curves for $\mu = 3.74$ and $\sigma = 0.002$ form a very tight envelope, while, for $\mu = 3.55$, the $\Lambda(k)$ curves show only a noiselike feature. Hence, we conclude that noise does induce chaos at $\mu = 3.74$, while noise-induced chaos does not happen in the main period(2)-doubling cascade.

Very interestingly, for $\mu = 3.63$ and 3.83, shown in Figs. 2(b) and 2(d), respectively, the linearly increasing segments of the $\Lambda(k)$ curves corresponding to two shells [shells $(2^{-(i+1)/2}, 2^{-i/2})$ with $i = 9, 10$ for $\mu = 3.63$, and $i = 7, 8$ for $\mu = 3.83$] also collapse together. These features indicate that the noisy dynamics at $\mu = 3.63$ and 3.83 appear to be chaoslike at certain definite scales. Note that Crutchfield *et al.* [4] also observed that the effect of noise on the period-doubling cascade associated with the period-3 window (about $\mu = 3.83$) is different from that associated with the main period(2)-doubling sequence in that the probability density of the former has a broad background.

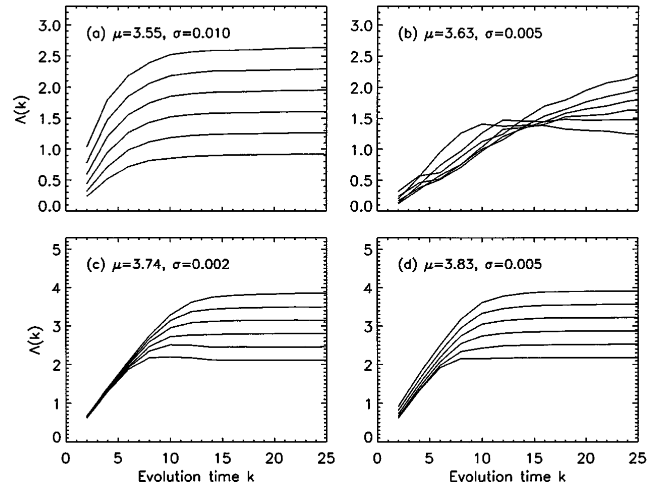

FIG. 2. Time-dependent exponent $\Lambda(k)$ vs evolution time k curves for (a) $\mu = 3.55$ and $\sigma = 0.01$; (b) $\mu = 3.63$ and $\sigma = 0.005$; (c) $\mu = 3.74$ and $\sigma = 0.002$; and (d) $\mu = 3.83$ and $\sigma = 0.005$. Six curves, from bottom to top [in terms of the $\Lambda(k)$ values for large k], correspond to shells $(2^{-(i+1)/2}, 2^{-i/2})$ with $i = 7, 8, 9, 10, 11$, and 12.

Before leaving Fig. 2, we note an unexpected feature exhibited by Fig. 2(b): The time-dependent exponent curves for $\mu = 3.63$ cross over for small evolution times. This reflects that the effect of noise on the system behavior is different at different scales. This may be related to the fact that the adjacent chaotic motions at about $\mu = 3.63$ are sensitive to noise.

Let us look further into why noise does or does not induce chaos at $\mu = 3.55$, 3.63, 3.74, and 3.83. For this purpose, we choose a series of different noise levels and study the behavior of the system at these parameter values.

For noisy oscillatory systems such as the noisy Van der Pol's oscillator, or the wakes behind a cylinder, we have shown [13] that points in the phase space execute Brownian-like motions, characterized by a power law growth of the logarithmic displacement curves: $\langle \ln \|X_{i+k} - X_{j+k}\| \rangle \sim \ln k^{0.5}$, for large k. We have also found in a semiconductor laser system [14] that, near a bifurcation point, the long-term growth rate of the displacement curves may be slowed down, characterized by a power law growth of $\langle \ln \|X_{i+k} - X_{j+k}\| \rangle \sim \ln k^{\alpha}$, with $\alpha < 0.5$. This is because the convergent flow of the underlying deterministic periodic orbit is very weak, hence noise can instantly kick phase points off the deterministic orbit to a region where nonlinearity is very strong, resulting in a diffusional process that is slower than the standard Brownian motion for large evolution times. With these results in mind, we also compute the logarithmic displacement curves for the noisy logistic map.

Figure 3 shows four groups of the logarithmic displacement $\langle \ln \|X_{i+k} - X_{j+k}\| \rangle$ curves for (a) $\mu = 3.74$ and $\sigma = 0.0003$, (b) $\mu = 3.83$ and $\sigma = 0.001$, (c) $\mu = 3.63$ and $\sigma = 0.0003$, and (d) $\mu = 3.55$ and $\sigma = 0.0005$. To separate these different groups of curves from each other, the groups (a) and (b) curves are shifted upward by 2 and 1 units, while the groups (c) and (d) curves are shifted downward by -0.5 and -0.2 units, respectively. Also shown in (a) (as diamonds), (b) (as triangles), (c) (as circles), and (d) (as squares) are curves generated from $\ln k^{\alpha}$ with $\alpha = 1.5$, 1.0, 1.0, and 0.25, respectively. Surprisingly, we have observed a new type of diffusional process, which is stronger than the standard Brownian motion characterized by an exponent $\alpha > 0.5$. How shall we understand this?

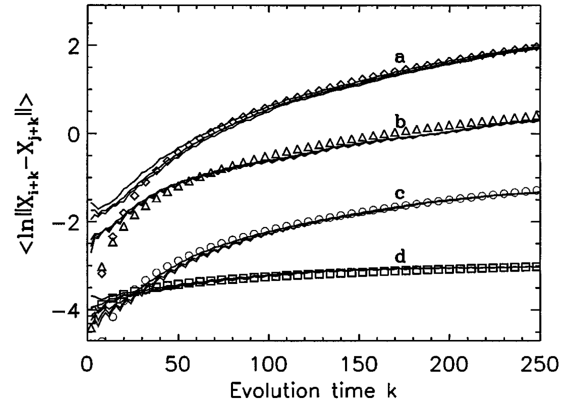

FIG. 3. Three logarithmic displacement $\langle \ln \|X_{i+k} - X_{j+k}\| \rangle$ curves (from top to bottom) corresponding to shells $(2^{-(i+1)/2}, 2^{-i/2})$ with $i = 12$, 13, and 14. For more details, see the text.

This is the very condition we seek for the periodic states themselves to be susceptible to noise-induced chaos. Deterministic chaos is characterized by short-term exponential divergence between nearby orbits. To induce chaos by adjusting the noise level, we are trying to make the displacement curves of the noisy system grow exponentially for a short period of time, then level off. This is much easier for noisy systems that already show diffusional processes stronger than the Brownian motion for very weak noise (such as $\sigma = 0.0003$ for $\mu = 3.74$). This is the reason that noise is able to induce chaos at $\mu = 3.63$, 3.74, and 3.83, while the chaoslike motion at $\mu = 3.74$ is much better defined than that at $\mu = 3.63$ and 3.83.

We are now ready to answer whether noise masks or inhibits a complete period-doubling cascade. The answer depends on whether noise-induced chaos occurs or not. When noise can induce a transition from a periodic state to a chaotic state, we conclude that a particular period-doubling cascade is inhibited. If noise cannot induce chaos, we anticipate that the period-doubling cascade is only masked by noise. To make the above idea more concrete, let us observe some noisy time series at $\mu = 3.55$ and 3.74. Figure 4 shows some typical time series for the clean and noisy systems with the noise strength

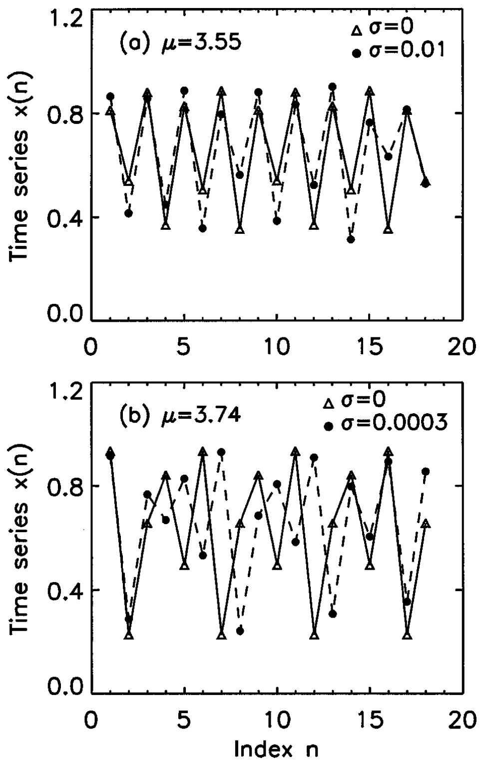

FIG. 4. Typical time series for (a) $\mu = 3.55$ and (b) $\mu = 3.74$. The points designated by open triangles and connected by solid curves are for the clean system, while the points designated by filled circles and connected by dashed lines are for the noisy system.

designated in the figure. We note that, for $\mu = 3.74$, even with a tiny amount of noise ($\sigma = 0.0003$), the noisy time series is already very different from the clean periodic time series. In contrast, for $\mu = 3.55$, even at a very high noise level ($\sigma = 0.01$), we still observe that the noisy time series follows closely the periodic pattern of the clean signal. Note that, by calculating the probability density of the noisy system, Crutchfield *et al.* [4] concluded that a period of 4 is the highest that remains in the primary 2^n cascade with a noise strength of $\sigma = 0.001$. By inspecting the time series, we observe, however, that even with $\sigma = 0.01$ the noisy system still follows closely the periodic pattern of the clean system. This is strong evidence that noise only masks the primary 2^n period-doubling cascade.

A complexity may arise for a particular period-doubling cascade, for example, the period(3)-doubling cascade at about $\mu = 3.83$. When the noise is so weak that chaos has not been induced yet, the bifurcation sequence may be simply masked by noise. When the noise is so strong that chaos has been induced (or even the induced chaos has been destroyed), then the difference between different periodic states is lost, and the period-doubling cascade is inhibited.

In summary, we have shown that noise can indeed induce chaos. Three basic ingredients are required for this to happen. First of all, the noise level has to fall within a certain narrow range. Noise below this range would not be sufficient to induce chaos, whereas noise above this range would destroy the induced chaos, if chaos can indeed be induced. Second, when subject to a noise source of strength within this range, the adjacent chaotic states should still behave chaotically on certain finite scales. Third, and most important, the periodic state itself, when subject to weak noise, should undergo a process that is much more diffusive than the Brownian motion. For a particular period-doubling cascade, when noise induces chaos, the complete period-doubling sequence is inhibited. Otherwise, the cascade is simply masked by noise. We note that these findings have also been observed in a semiconductor laser system [15].

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