Statistical properties of multiplicative multifractal processes in modelling telecommunications traffic streams

J. Gao and I. Rubin

Telecommunications network systems have been observed to contain processing and switching nodes that are loaded by highly bursty traffic streams. Such flows exhibit long-range-dependent (LRD) properties. The design and sizing of electronic processing and switching systems depend critically on the features of such processes. LRD traffic streams are modelled as a multiplicative multifractal process. Through simulation, it is shown that similar queueing performance results can be simultaneously obtained for a network operating at low, medium, and high utilisation levels when one compares the system-size tail performance of a single server queueing system driven in the first case by the measured traffic and in the second case by a multiplicative multifractal traffic model. A number of the properties of such processes that are most relevant to their use as network traffic models are proved.

Introduction: Recent analysis of high-quality traffic measurements have revealed the prevalence of long-range-dependent (LRD) features in traffic processes loading packet switching communications networks. Included are local area networks (LANs) [1], wide area networks (WANs) [2], variable-bit rate (VBR) video traffic [3], and world wide web (WWW) traffic [4]. With the prevalence of LRD traffic flows in data networks, parsimonious modelling of such traffic has become important. It is essential to have a relatively simple model for the effective design and sizing of telecommunications buffering, switching and processing systems.

Among the many LRD traffic models published in the literature, two types of multiplicative multifractal models have distinguished themselves [5, 6]. One type involves modelling interarrival time series and packet length sequences using two multifractals [5]. Another type involves first choosing a suitable fixed size time interval, and then modelling the associated counting process using a multiplicative multifractal [6]. These models yield similar queueing performance results simultaneously for a network operating at low, medium, and high utilisation levels when one compares the system-size tail performance of a single server queueing system driven in the first case by the measured traffic and in the second case by a multiplicative multifractal traffic model. The measured traffic streams examined in [5, 6] include flows across local and metropolitan area networks, involving a multitude of applications such as VBR video and WWW oriented services.

A quantitative understanding of why a multiplicative multifractal process makes a good LRD traffic model is, however, still lacking. In this Letter, we identify and prove a number of properties of such a process that are most relevant to their use as models for communications network traffic processes.

Definition of multifractal: We consider a unit interval and associate it with a unit mass. We partition the unit interval into a series of small intervals, each of linear length ϵ and also partition the unit mass into a series of weights $\{w_i\}$, and associate w_i with the ith interval. We now consider the moments

$$M_q(\epsilon) = \sum_i w_i^q \qquad (1)$$

where q is real. If we have, for a real function $\tau(q)$ of q

$$M_q(\epsilon) \simeq \epsilon^{\tau(q)} \quad \text{as } \epsilon \to 0 \qquad (2)$$

for every q, and the weights $\{w_i\}$ are nonuniform, then the weights $w_i(\epsilon)$ are said to form a multifractal measure. Note that the normalisation $\Sigma_i w_i = 1$ implies that $\tau(1) = 0$.

Construction of multiplicative multifractal: We consider a unit interval and associate it with a unit mass. We divide the unit interval into two segments of equal length and also partition the associated mass into two fractions, r and $1 - r$, and assign them to the left and right segments, respectively. The parameter r is in general a random variable, governed by a probability density function (pdf) $P(r)$, $0 \le r \le 1$. The fraction r is called the multiplier. Each new subinterval and its associated weight are further divided into two parts following the same rule. This procedure is shown schematically in Fig. 1, where the multiplier r is written as r_{ij}, with i indicating the stage number. Note the scale (i.e. the interval length) associated with stage i is 2^{-i}. We assume that $P(r)$ is symmetric about $r = 1/2$, and has successive moments $\mu_1, \mu_2, \ldots$ Hence r_{ij} and $1 - r_{ij}$ both have a marginal distribution $P(r)$. The weights at the stage N, $\{w_n, n = 1, \ldots, 2N\}$, can be expressed as $w_n = u_1 u_2 \cdots u_N$, where u_l, $l = 1, \ldots, N$, are either r_{ij} or $1 - r_{ij}$. Thus, $\{u_i, i \ge 1\}$ are independent identically distributed random variables having a pdf $P(r)$.

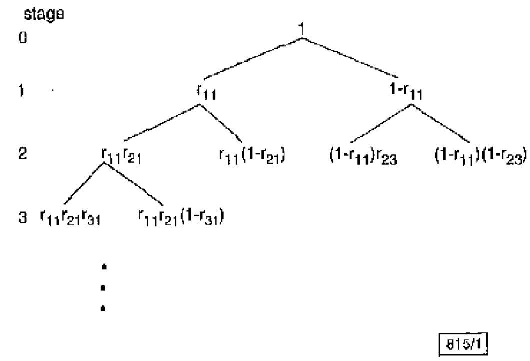

Fig. 1 *Schematic diagram of construction rule of multiplicative multifractal*

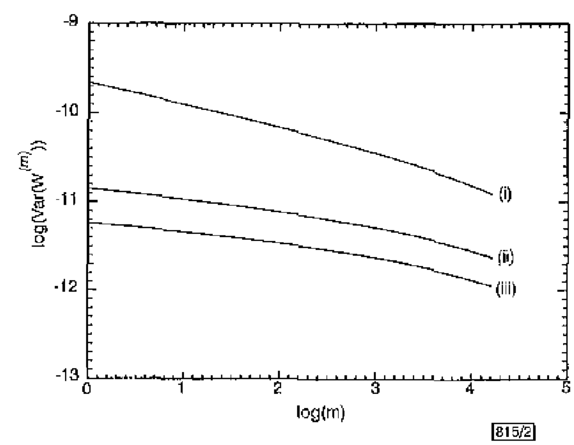

Fig. 2 *$\log(Var(W^{(m)}))$ against $\log(m)$ curves*

(i) $\alpha = 10$
(ii) $\alpha = 50$
(iii) $\alpha = 100$

Properties of multiplicative multifractals: For the weights at stage N, we prove that the following properties hold:

(i) $M_q(\epsilon) \simeq \epsilon^{\tau(q)}$, with $\varepsilon = 2^{-N}$, $\tau(q) = -\ln(2\mu_q)/\ln 2$. This follows the observation that at stage N, $M_q(\epsilon) = E(\Sigma_{n=1}^{2^N} (w_n)^q) = 2^N E(w^q) = 2^N E((u_1 u_2 \cdots u_N)^q) = 2^N \mu_q^N$. This property indicates that a multiplicative process is a multifractal, and relates the $\tau(q)$ spectrum to the moments of the multiplier distribution.

(ii) $E(w) = E(w_n) = E(u_1 u_2 \cdots u_N) = 2^{-N}$, $n = 1, \ldots, 2^N$.

(iii) $Var(w) = Var(w_n) = \mu_2^N - 2^{-2N}$, $n = 1, \ldots, 2^N$. To prove this relation, we note that $E(w^2) = E(w_n^2) = E((u_1 u_2 \cdots u_N)^2) = \mu_2^N$.

(iv) When $N \gg 1$, the weights at stage N have a log-normal distribution. This is deduced directly by taking the logarithm of $w_n = u_1 u_2 \cdots u_N$.

(v) $E[(w_n - E(W))(W_{n+m} - E(w))] = (1/2 - \mu_2)\mu_2^{N-1}(4\mu_2)^{-k} - 2^{-2N}$, for $m = 2^k$, where k is an integer. Hence, the covariance function decays with time lag m in a power-law manner.

To prove assertion (v), we consider two weights w_{n_1} and w_{n_2} at stage N. Assume they share the same ancestor weight x at stage $N - k$, i.e. $w_{n_1} = x\, r \Pi_{l=1}^{k-1} r_{1l}$, $w_{n_2} = x(1 - r)\Pi_{l=1}^{k-1} r_{2l}$, where r and $\{r_{il}, i = 1, 2, l = 1, \ldots, k - 1\}$ are independent random variables with distribution $P(r)$. Then $E[(w_{n_1} - 2^{-N})(w_{n_2} - 2^{-N})] = E(x^2)\, E[r(1 - r)]\, E[\Pi_{l=1}^{k-1} r_{1l} r_{2l}] - 2^{-2N} = 2^{-2(k-1)} \mu_2^{N-k} (1/2 - \mu_2) - 2^{-2N}$. For $m = 2^k$, all pairs of $\{w_n, w_{n+m}\}$, for $n \ge 1$, share an ancestor at stage $N - k - 1$. Hence, $E[(w_n - E(w))(w_{n+m} - E(w))] = 2^{-2k}\mu_2^{N-k-1} (1/2 - \mu_2) - 2^{-2N} = (1/2 - \mu_2)\, \mu_2^{N-1} (4\mu_2)^{-k} - 2^{-2N}$.

(vi) $Var(W^{(m)}) = \mu_2^N (4\mu_2)^{-k} - 2^{-2N}$, where $W^{(m)} = (w_{im-m+1} + \cdots + w_{im})/m$, $m = 2^k$, $k = 1, 2, \ldots$, and $i \geq 1$. This is proven by expressing $W^{(m)} = 2^{-k}x$, where x is a weight at stage $N - k$.

The equation in (vi) expresses a variance-time relation. For LRD traffic [1], $Var(W^{(m)}) = m^{2H-2}$, where $1/2 < H < 1$ is the Hurst parameter. For multiplicative multifractal processes, when N is large and $\mu_2 > 0$, the term $\mu_2^N (4\mu_2)^{-k}$ dominates. Hence, $\log(Var(W^{(m)}))$ against $\log(m)$ is linear, with the resulting slope, $-\log(4\mu_2)/\log 2$, providing an estimate of $2H - 2$. Several examples of the variance-time curves are shown in Fig. 2 with $P(r) = e^{-\alpha(r-1/2)^2}$, $0 \leq r \leq 1$, where α is a parameter, and $N = 18$.

Also note that the dependence of $Var(W^{(m)})$ on m is the same as that of $E[(w_n - E(w))(w_{n+m} - E(w))]$ on m.

Conclusions: We have proved a number of properties for multiplicative multifractal processes. These properties will help in the modelling of LRD telecommunications network traffic processes by using multiplicative multifractal processes.

Acknowledgment: This work is supported by UC MICRO/SBC Pacific Bell research grant 98-131 and by ARO grant DAAGJJ-98-1-0338.

13 October 1999
Electronics Letters Online No: 20000133
DOI: 10.1049/el:20000133

J. Gao and I. Rubin (*Department of Electrical Engineering, University of California at Los Angeles, Los Angeles, CA 90095, USA*)

References

1 LELAND, W.E., TAQQU, M.S., WILLINGER, W., and WILSON, D.V.: 'On the self-similar nature of ethernet traffic (extended version)', *IEEE/ACM Trans. Netw.*, 1994, **2**, pp. 1–15

2 PAXSON, V., and FLOYD, S.: 'Wide area traffic – the failure of Poisson modeling', *IEEE/ACM Trans. Netw.*, 1995, **3**, pp. 226–244

3 BERAN, J., SHERMAN, R., TAQQU, M.S., and WILLINGER, W.: 'Long-range-dependence in variable-bit-rate video traffic', *IEEE Trans. Commun.*, 1995, **43**, pp. 1566–1579

4 CROVELLA, M.E., and BESTAVROS, A.: 'Self-similarity in World Wide Web traffic: Evidence and possible causes', *IEEE/ACM Trans. Netw.*, 1997, **5**, pp. 835–846

5 GAO, J.B., and RUBIN, I.: 'Multiplicative multifractal modeling of long-range-dependent traffic'. Proc. ICC'99, June 1999, (Vancouver, Canada)

6 GAO, J.B., and RUBIN, I.: 'Multifractal modeling of counting processes of long-range dependent network traffic'. Proc. SCS Advanced Simulation Technologies Conf., San Diego, CA, 1999

ESTIMATION OF LENGTH FOR *ELECTRONICS LETTERS* VERSION 5.1

As set by our typesetters, the maximum length in print of a Letter is 83 column cms, or three columns of about 27cms height. When judging the length of a submission, please take into account:

Title 0.5 column cm per 40 characters (or part thereof)

Abstract 1 column cm per 165 characters

Authors 1 column cm per separate affiliation

Text 1 column cm per 165 characters (it is easiest to estimate the number of characters, including spaces, per line and the number of lines per page)

References 1 column cm each

Tables 0.4 column cm per line

Figures All figures are reduced to a width of 8.6cm. If the figure is wider than it is high, divide the height by the width and multiply by 8.6 column cm. If the figure is higher than it is wide (discouraged), the figure will occupy more than 8.6 column cm.

Captions 1 column cm per main figure or table caption (provided it is brief), 0.33 column cm per line for subcaptions (including keys and other information that will be removed from the figures).

Equations

Single line equations:	0.6–0.8	column cm per line
Integrals:	1.0	column cm per line
Quotients:	0.7–1.2	column cm per line
Sums and products:	1.2	column cm per line
Matrices:	0.4	column cm per line

For other equation types, refer to a recent issue of *Electronics Letters*. Excessively long equations will be split into several lines.