Superposition of Multiplicative Multifractal Traffic Streams

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Abstract: Source traffic streams as well as aggregated traffic flows often exhibit long-range-dependent (LRD) properties. In this paper, we model each traffic stream component through the multiplicative multifractal counting process traffic model. We prove that the superposition of a finite number of multiplicative multifractal traffic streams results in another multifractal stream. This property makes the multifractal traffic model a versatile tool in modeling traffic streams in computer communication networks. There, a node is loaded by a traffic flow resulting from the superposition of source streams and aggregated LRD (and other) streams. The structure and the burstiness of the superimposed process is studied, and useful mathematical relations are obtained.

1 Introduction

Recent analysis of high-quality traffic measurements have revealed the prevalence of long-range-dependent (LRD) features in traffic processes loading packet switching communications networks. Included are local area networks (LANs) [8], wide area networks (WANs) [9], variable-bit-rate (VBR) video traffic [1,7], and world wide web (WWW) traffic [2]. With the prevalence of LRD traffic flows in data networks, the modeling of such traffic has become important.

It has been shown that a multiplicative multifractal traffic model can yield similar queueing performance results simultaneously for a network operating at low, medium, and high utilization levels when one compares the system-size tail performance of a single server queueing system driven on one hand by a multiplicative multifractal traffic model and on the other hand by the measured traffic streams such as flows across local and metropolitan area networks, involving a multitude of applications such as VBR video and WWW oriented services [3-5]. Quantitative understanding on why a multiplicative multifractal process makes a good LRD traffic model has also been obtained [6]. In communication networks, the traffic process loading nodal switching and transmission processors is noted to be described as the superposition of multiple input traffic streams. In this paper, we use the counting process model [4] to prove that superposition of multiplicative multifractal traffic streams results in another multifractal stream. This property allows us to model LRD traffic flows at different network locations using a single multiplicative multifractal counting process model. In particular, when there are a number of independent users each generating a LRD source traffic modeled by a multiplicative multifractal, in so far as the aggregated traffic is concerned, one need only to simulate one multiplicative multifractal for the aggregated traffic instead of simulating a bunch of multifractals for all the users.

The remaining of the paper is organized as follows. In Sec. 2 we review briefly the counting multiplicative multifractal traffic process model. In Sec. 3, we prove that the superposition of a finite number of multiplicative multifractals results in another multifractal. The burstiness of the superimposed process is studied in Sec. 4. Conclusions are given in Sec. 5.

2 Multiplicative multifractal counting process traffic model

Consider a unit interval. Associate it with a unit mass. Divide the unit interval into two (say, left and right) segments of equal length. Also partition the mass into two fractions, r and $1-r$, and assign them to the left and right segments, respectively. The parameter r, called the multiplier, is in general a random variable, governed by a probability density function (pdf) $P(r)$, $0 \le r \le 1$. Each new subinterval and its associated weight (or mass) are further divided into two parts following the same rule. This procedure is schematically shown in Fig. 1, where the multiplier r is written as r_{ij}, with i indicating the stage number. Note the scale (i.e., the interval length) associated with stage i is 2^{-i}. We assume that $P(r)$ is symmetric about $r = 1/2$, and has successive moments $\mu_1, \mu_2, \ldots$. Hence, the weights at the stage N, $\{w_n(N), n = 1, \ldots, 2^N\}$, can be expressed as $w_n(N) = u_1 u_2 \cdots u_N$, where u_l, $l = 1, \ldots, N$, are independent identically distributed (i.i.d) random variables having pdf $P(r)$. When $w_n(N)$ is interpreted as the loading to a network (representing the total count

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of message units) in a time slot of length $2^{-N}T$, where T is the total time period one is interested in, then this process becomes a counting traffic process model. The multifractality of the multiplicative process refers to the fact that $M_q(\epsilon) = E(\sum_{n=1}^{2^N}(w_n(N))^q) \sim \epsilon^{\tau(q)}$, with $\epsilon = 2^{-N}$, $\tau(q) = -\ln(2\mu_q)/\ln 2$ [6].

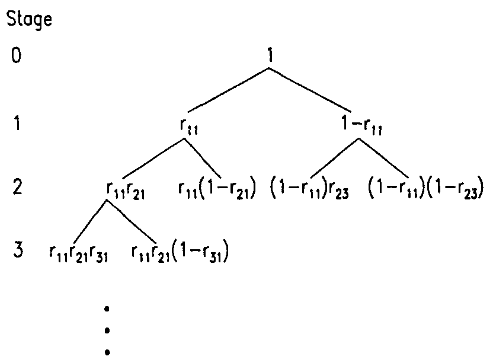

Figure 1: Schematic illustrating the construction rule of a multiplicative multifractal.

3 Superposition of multiplicative multifractals

Consider the superposition of (an arbitrary) k independent multiplicative multifractal traffic streams. Let these multifractal traffic streams be denoted as MF_1, ..., MF_k. Their multiplier distributions are $P^{(1)}(r), ..., P^{(k)}(r)$. These distributions are assumed to be symmetric about 1/2, and have successive moments $\mu_q^{(i)}$, $i = 1, ..., k$, $q = 1, 2, \cdots$. The superimposed traffic stream is denoted by $MF^{(sk)}$, $MF^{(sk)} = \sum_{i=1}^k \lambda_i \cdot MF_i$, with $0 < \lambda_1, ..., \lambda_k < 1$, $\sum_{i=1}^k \lambda_i = 1$. A weight $w^{(sk)}(N)$ of $MF^{(sk)}$ at the stage N can then be expressed as $w^{(sk)}(N) = \sum_{i=1}^k \lambda_i w^{(i)}(N) = \sum_{i=1}^k \lambda_i u_1^{(i)} \cdots u_N^{(i)}$, where $w^{(i)}(N)$ is a weight of MF_i at stage N, and $u_j^{(i)}$, $j = 1, ..., N$ are i.i.d random variables governed by pdf $P^{(i)}(r)$, for $i = 1, ..., k$.

We first prove the following simple properties for the weights of $MF^{(sk)}$ at stage N.

(i) $E(w^{(sk)}(N)) = E(w_i^{(sk)}(N)) = \sum_{i=1}^k \lambda_i E(w^{(i)}(N)) = 2^{-N}$, $i = 1, ..., 2^N$.

(ii) Since MF_1, ..., MF_k are independent, we have $Var(w^{(sk)}(N)) = Var(w_i^{(sk)}(N)) = \sum_{j=1}^k \lambda_j^2 Var(w^{(j)}(N))$, $i = 1, ..., 2^N$.

We can now estimate the Hurst parameter for $MF^{(sk)}$ by considering the variance-time relation. Let $W^{(m)} = (w_{im-m+1}^{(sk)} + \cdots + w_{im}^{(sk)})/m$, $W_j^{(m)} = (w_{im-m+1}^{(j)} + \cdots + w_{im}^{(j)})/m$, $j = 1, ..., k$, then we have $Var(W^{(m)}) = \sum_{j=1}^K \lambda_j^2 Var(W_j^{(m)})$, where $Var(W_j^{(m)}) = (\mu_2^{(j)})^N (4\mu_2^{(j)})^{-k} - 2^{-2N}$, $j = 1, ..., k$ [6]. Since each term in $Var(W^{(m)})$ is an exponential term, for reasonably large N, only the one corresponding to the maximum of $\mu_2^{(j)}$, $j = 1, ..., k$, dominates. Dropping other terms, we then obtain a power-law relation between $Var(W^{(m)})$ and m. We thus find the Hurst parameter $H^{(sk)}$ for $MF^{(sk)}$ $H^{(sk)} \approx min(H^{(1)}, \cdots, H^{(k)})$, where $H^{(i)}$, $i = 1, \cdots, k$, are the Hurst parameters associated with $MF^{(i)}$, $i = 1, \cdots, k$. We thus observe a sharp difference between the superposition of multifractal traffic and superposition of other LRD traffic models such as fractional Brownian motion processes or the heavy-tailed ON/OFF models. In the latter situations, the superimposed process assumes a value for its Hurst parameter which is the maximum of the source traffic streams [10].

(iii) $E[(w^{(sk)}(N))^q] = E[(\sum_{i=1}^k \lambda_i u_1^{(i)} \cdots u_N^{(i)})^q] = l[\mu_q^{(sk)}]^N[1 + \sum_{j=0}^{q'} b_j y_j^N]$, where l, $\mu_q^{(sk)}$, $\{b_j,\ y_j,\ |y_j| < 1,\ j = 0, ..., q'\}$, and q' are suitable constants.

Proof. $E[(\sum_{i=1}^k \lambda_i u_1^{(i)} \cdots u_N^{(i)})^q]$
$= E[\sum \frac{q!}{q_1! \cdots q_k!}(\lambda_1 u_1^{(1)} \cdots u_N^{(1)})^{q_1} \cdots (\lambda_k u_1^{(k)} \cdots u_N^{(k)})^{q_k}]$
$= \sum \frac{q!}{q_1! \cdots q_k!}[\lambda_1^{q_1}(\mu_{q_1}^{(1)})^N] \cdots [\lambda_k^{q_k}(\mu_{q_k}^{(k)})^N]$, where $\sum_{i=1}^k q_i = q$. Let

$$\mu_q^{(sk)} = \max_{q_1, \ldots, q_k} (\mu_{q_1}^{(1)} \cdots \mu_{q_k}^{(k)}). \tag{1}$$

Assume among all the terms $\mu_{q_1}^{(1)} \cdots \mu_{q_k}^{(k)}$, there are m terms that attain the above maximal value. Group those terms together. We can then write $E[(w^{(sk)}(N))^q] = l[\mu_q^{(sk)}]^N[1 + \sum_{j=0}^{q'} b_j y_j^N]$, with $l = l(m)$, $\mu_q^{(sk)}$, $\{b_j = b_j(m),\ y_j,\ |y_j| < 1,\ j = 0, ..., q'\}$, and $q' = q'(m)$ are suitable constants. This concludes the proof.

From the above result, we immediately have $E[(w^{(sk)})^q] \sim \mu_q^N$, $N \to \infty$.

We are now ready to prove the following theorem.

Theorem 1: $MF^{(sk)}$ **is (asymptotically in** N**) a multifractal.**

The proof is quite straightforward if one notices that $M_q(\epsilon) = 2^N E[(w^{(sk)}(N))^q] \sim \epsilon^{\tau(q)}$, with $\epsilon = 2^{-N}$, $\tau(q) = -\ln(2\mu_q^{(sk)})/\ln 2$, when $N \to \infty$.

We can prove that the moments of the multiplier distribution of $MF^{(sk)}$ are (asymptotically in N) stage-independent. Hence, one expects $MF^{(sk)}$ to be for most practical situations a multiplicative multifractal.

Indeed, we have observed in our numerical simulations that $MF^{(sk)}$ is asymptotically a multiplicative process. To illustrate this feature, we present an example involving the superposition of 2 multiplicative multifractal traffic streams, MF_1 and MF_2, with the largest stage number being 18. Let $\lambda_1 = \lambda_2 = 1/2$, and the multiplier distributions for MF_1 and MF_2 be truncated Gaussian: $P(r) \sim e^{-\alpha(r-1/2)^2}$, $0 \leq r \leq 1$, with $\alpha = 50$ and 100 for MF_1 and MF_2, respectively. To estimate the transformation between $w^{(s2)}(N+1)$ and $w^{(s2)}(N)$, we use

$$r_i(N) = \frac{w_{2i-1}^{(s2)}(N+1)}{w_i^{(s2)}(N)}, \quad i = 1, ..., 2^N, \qquad (2)$$

where $w_i^{(s2)}(N)$ and $w_{2i-1}^{(s2)}(N+1)$, $i = 1, ..., 2^N$, are the weights of $MF^{(s2)}$ at stages N and $N+1$, respectively. We compute the distribution $P_N(r)$ from its histogram based on $\{r_i(N), \quad i = 1, ..., 2^N\}$. We then plot $P_N(r)$ vs. r for different stages N. Fig. 2 shows $P_N(r)$ vs. r curves for $N = 13, ..., 17$.

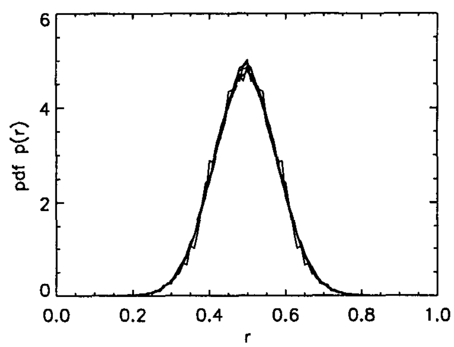

Figure 2: Multiplier distributions $P(r)$ vs. r curves for the stage numbers $N = 13, ...17$.

We observe that those curves neatly collapse together, indicating that $P_N(r)$ is quite independent of the stage number N for reasonably large values of N.

Next we model $MF^{(s2)}$ by a single ideal multiplicative multifractal, $MF^{(e)}$. For this purpose, we simply use the distribution $P_N(r)$ as calculated for reasonably large values of N as the multiplier distribution for $MF^{(e)}$. From Fig. 2 we find that $P_N(r)$ can be well fitted by a truncated Gaussian distribution with $\alpha = 67$. We then generate $MF^{(e)}$ till stage $N = 18$. We drive a single server queueing system, on one hand, by $MF^{(e)}$, and on the other hand, by $MF^{(s2)}$, and compare the system size tail distributions. Fig. 3 shows this comparison. We observe that the system-size tail distributions are almost identical for the two queueing systems for all the four utilization levels, $\rho = 0.3$, 0.5, 0.7 and 0.9. Hence, $MF^{(s2)}$ is indeed equivalent to $MF^{(e)}$ in terms of queueing performance.

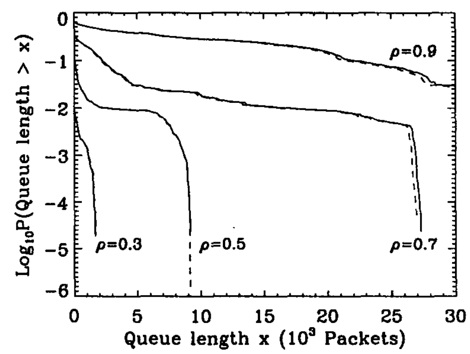

Figure 3: System-size tail distributions obtained when $MF^{(s2)}$ (dashed lines) and MF^e (solid lines) are used to drive identical single server queueing systems. The utilization levels are indicated on the figure.

4 Burstiness of the superimposed process

In this section, we study the burstiness of the superimposed multifractal traffic processes. A traffic process A is said to be more bursty than a traffic process B if a single server queueing system yields a longer system size tail distribution when the process A is used to drive the queueing system. For ease of exposition, we shall only consider superposition of two traffic streams. Furthermore, we assume that $\lambda_1 = \lambda_2 = 1/2$, each source traffic stream has 2^{18} counting states, and all source traffic streams have the same mean.

We consider three cases: (i) Superposition of two homogeneous sources. That is, MF_1 and MF_2 are two different realizations of a multiplicative multifractal process with a given multiplier distribution. (ii) Superposition of two heterogeneous sources. That is, MF_1 and MF_2 have different multiplier distributions $\{P(r)\}$. (iii) Superposition of a multiplicative multifractal process with a Poisson process. We shall show that in case (i), $MF^{(s2)}$ is always less bursty than either MF_1 or MF_2. In case (ii), if we assume MF_1 to be more bursty than MF_2, then $MF^{(s2)}$ is always less bursty than MF_1, but can be more bursty than MF_2. In case (iii), $MF^{(s2)}$ is less bursty than the multifractal traffic component, while more bursty than the Poisson process. The latter can actually be substituted by a deterministic process. Note the latter to be a particular multiplicative multifractal process with $P(r) = \delta(r - 1/2)$.

A good starting point for the study of cases (i) and (ii) is to employ the variance-time relation obtained in the last section. Without loss of generality, we assume that MF_1 is at least as bursty as MF_2. When the multiplier distributions $\{P(r)\}$ for MF_1 and MF_2 are of the same functional form (but with different parameters), we conclude the following inequality, $Var(W_1^{(m)}) \geq Var(W_2^{(m)})$ [3,4]. We then have $Var(W^{(m)}) < Var(W_1^{(m)})$. This inequality motivates our observation that $MF^{(s2)}$ is less bursty than MF_1. This is verified to be the case by the following numerical examples (as well as other cases examined by us and not presented here).

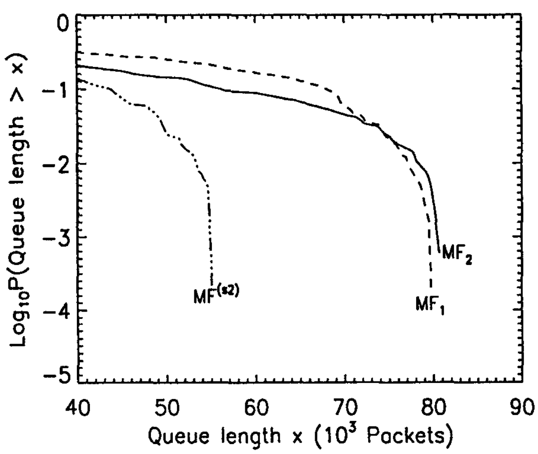

Figure 4: System-size tail distributions obtained when $MF^{(s2)}$, MF_1 and MF_2 are used to drive identical single server queueing systems. The utilization level is 0.9.

We examine first case (i) involving the superposition of two homogeneous sources. We consider two realizations of a multiplicative process with its $P(r)$ function given by

$$P(r) = \begin{cases} q + p\delta(r - 1/2) & 1/2 - d \leq r \leq 1/2 + d \\ 0 & \text{otherwise} \end{cases}$$

with $(p, d) = (0.66, 0.3)$. The system-size tail distributions for queueing systems operating at utilization level $\rho = 0.9$ when MF_1, MF_2, and $MF^{(s2)}$ are used to drive them are shown in Fig. 4. We observe that $MF^{(s2)}$ is clearly less bursty than either MF_1 or MF_2.

For case (ii) involving the superposition of two heterogeneous sources, let $P(r)$ be given by truncated Gaussian with $\alpha = 50$ for MF_1 and $\alpha = 100$ for MF_2. We generate one realization for MF_1, and 2 realizations for MF_2, and consider superposition of the realization of MF_1 with those of MF_2. The system-size tail distributions for queueing systems operating at utilization level $\rho = 0.7$ when MF_1, MF_2, and $MF^{(s2)}$ are used to drive them are shown in Fig. 5.

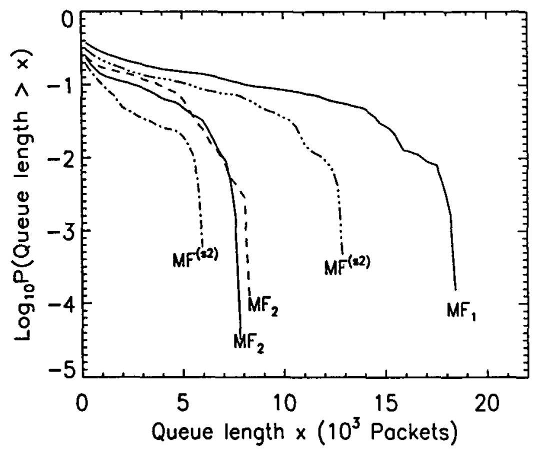

Figure 5: System-size tail distributions obtained when $MF^{(s2)}$, MF_1 and MF_2 are used to drive identical single server queueing systems. The utilization level is 0.7.

We observe that while $MF^{(s2)}$ is always less bursty than MF_1, it can be both more bursty or less bursty than MF_2. Closer examination reveals that when $MF^{(s2)}$ is more bursty than MF_2, the superposition of MF_1 and MF_2 is in phase, i.e., at the first few stages (corresponding to long time scales), large weights of MF_1 are added to large weights of MF_2. When $MF^{(s2)}$ is less bursty than MF_2, the superposition is out of phase, i.e., large weights of MF_1 at the first several stages are added to small weights of MF_2.

Finally, we examine case (iii) involving the superposition of a multiplicative process and a Poisson process. Since a Poisson traffic process is much much less bursty than a multiplicative multifractal traffic process, we expect that it can be effectively approximated by a deterministic process. Let our MF_1 be the same as that discussed in case (i). Denote the superposition of MF_1 and a Poisson process by $MF^{(1+p)}$, and the superposition of MF_1 and a deterministic process by $MF^{(1+d)}$. We then drive a single server queueing system by $MF^{(1+p)}$ and $MF^{(1+d)}$, and compute the system size tail distributions under different utilization levels. The results are shown in Fig. 6. Clearly we see that $MF^{(1+p)}$ is equivalent to $MF^{(1+d)}$. In other words, a Poisson process can be substituted by the deterministic process.

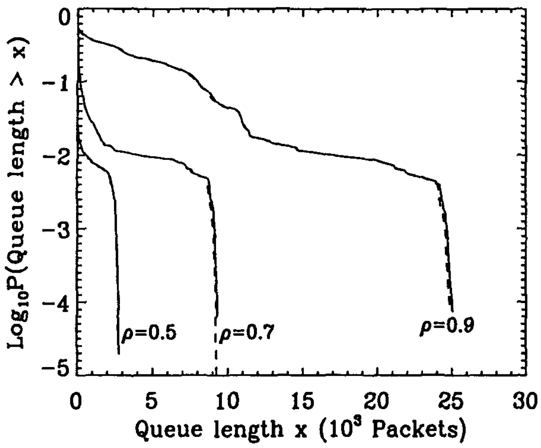

Figure 6: *System-size tail distributions obtained when* $MF^{(1+p)}$ (solid curves) and $MF^{(1+d)}$ (dashed lines) are used to drive a queueing system.

5 Conclusions

We have studied the superposition of multiplicative multifractal counting processes. We have proved that the superposition of an arbitrary finite number of multiplicative multifractal traffic streams results (asymptotically) in another multifractal. The Hurst parameter for the superimposed process is the same as the corresponding one for the source traffic stream that has the largest second moment of the multiplier distribution. Furthermore, we find in numerical simulations that the superimposed process is typically asymptotically a multiplicative process, and can be modeled by a single ideal multiplicative multifractal. These properties ensure that traffic streams representing LRD source traffic as well as LRD aggregated traffic in a communications network can be characterized by a single convenient model. In particular, these results shed light on why aggregated LAN and WAN traffic streams can be effectively represented as an ideal multiplicative multifractal traffic stream [3,4]. We have also examined the burstiness of the superimposed traffic streams by measuring the tail distributions they induce when applied to a single server queueing system. By examining a wide range of LRD traffic cases, we demonstrate that the superimposed process is less bursty than the most bursty traffic component. We also note that when a Poisson process is superimposed with a bursty multiplicative multifractal traffic (as a nonnegligible component), the Poisson component can be effectively replaced by a deterministic process in deriving a superimposed process that provides the same queueing tail features as those exhibited by the original process.

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