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DIFFUSION, INTERMITTENCY, AND NOISE-SUSTAINED METASTABLE CHAOS IN THE LORENZ EQUATIONS: EFFECTS OF NOISE ON MULTISTABILITY

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Multistability is an interesting phenomenon of nonlinear dynamical systems. To gain insights into the effects of noise on multistability, we consider the parameter region of the Lorenz equations that admits two stable fixed point attractors, two unstable periodic solutions, and a metastable chaotic "attractor". Depending on the values of the parameters, we observe and characterize three interesting dynamical behaviors: (i) noise induces oscillatory motions with a well-defined period, a phenomenon similar to stochastic resonance but without a weak periodic forcing; (ii) noise annihilates the two stable fixed point solutions, leaving the originally transient metastable chaos the only observable; and (iii) noise induces hopping between one of the fixed point solutions and the metastable chaos, a three-state intermittency phenomenon.

Keywords: Diffusion; intermittency; multistability.

1. Introduction

Multistability is an interesting phenomenon that has been observed in a variety of nonlinear systems including electronic circuits [Losson *et al.*, 1993], fluids experiments [Maurer & Libchaber, 1980], lasers [Brun *et al.*, 1985; Tredicce *et al.*, 1986; Dangoisse *et al.*, 1987; Solari *et al.*, 1987; Goswami, 1998; Pisarchik *et al.*, 2006; Martinez-Zerega & Pisarchik, 2005; Pisarchik *et al.*, 2003; Pisarchik & Pinto-Robledo, 2002; Pisarchik & Goswami, 2000], geophysical models [Kapitaniak *et al.*, 1995], mechanical systems [Thompson & Stewart, 1986; Freitas *et al.*, 2004], and in biological systems such as neurons [Foss *et al.*, 1996], human proprioception [Cordo *et al.*, 1996], and ambiguous visual perception [Simonotto *et al.*, 1997; Gao *et al.*, 2006a; Zhou *et al.*, 2004]. These multistable systems — like all natural and man-made systems — are immersed in and susceptible to noise, which can affect their multistable dynamics. Insights into noise-perturbed multistability could be exploited in the design or operation of such systems.

It is generally thought that near critical events such as bifurcations, a dynamical system may be very susceptible to small scale internal or external random fluctuations. Such considerations have motivated a number of interesting studies on the effects of noise on bifurcations and noise-induced transitions. Works along this line include study of the effects of noise on bifurcation parameters, as treated systematically by the monograph of [Arnold, 1998]; the effects of noise on Hopf and pitchfork bifurcations [Hoffman, 1982; Juel *et al.*, 1997]; the effects of noise on period-doubling bifurcations [Crutchfield *et al.*, 1982; Neiman *et al.*, 1994]; noise induced-hopping between two periodic states [Arecchi *et al.*, 1985]; noise-induced or inhibited hopping between a periodic state and a metastable chaotic state in a model equation of rf-biased Josephson junction [Kautz, 1985]; noise-induced hopping between chaotic states or two-state on-off intermittency in Duffing oscillators [Reategui & Pisarchik, 2004]; noise-induced periodicity (a phenomenon known as coherence resonance) [Pikovsky & Kurths, 1997]; noise-induced chaos [Crutchfield *et al.*, 1980; Crutchfield *et al.*, 1982; Kautz, 1985; Arecchi *et al.*, 1985; Gao *et al.*, 1999a; Gao *et al.*, 1999b; Hwang *et al.*, 2000]; and noise-induced Hopf bifurcation-type sequence and transitions to chaos [Gao *et al.*, 2002]. Here, we consider the effects of noise on the multistabiliy of the Lorenz equations. Specifically, we focus on the parameter region for the Lorenz equations that admits two stable fixed point attractors, two unstable periodic solutions, and a metastable chaotic "attractor". We describe and characterize three interesting dynamical behaviors: (i) noise-induced oscillatory motions, (ii) noise-induced and sustained metastable chaos, and (iii) switching between one of the fixed point solutions and the metastable chaos, a phenomenon of three-state intermittency.

2. Effects of Noise on a Multistability Region of the Lorenz System

We study the following noise-driven Lorenz equations

$$\frac{dx}{dt} = -\sigma(x - y),$$
$$\frac{dy}{dt} = rx - y - xz + D\eta(t), \qquad (1)$$
$$\frac{dz}{dt} = xy - bz,$$

where $D\eta(t)$ is a white Gaussian noise term with mean 0 and variance D^2, $\sigma = 10$, and $b = 8/3$. The noise term accounts for the fluctuations in the temperature difference between the ascending and descending currents, which is proportional to y. The SDE is solved using the scheme of Exact propagator [Mannella, 2002], where the exact solution of the Lorenz system is solved using a fourth order Runge–Kutta method with a time-step of $h = 0.002$, and then a term $D\sqrt{h}W$, where W is a Gaussian noise of mean 0 and variance 1, is added to the y-equation to take into account the noise.

When the noise is absent, for $r \in (24.06, 24.74)$, the system has two stable fixed points

$$C_+ = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1),$$
$$C_- = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1),$$

and a strange attractor. Depending on initial conditions, the solution will lead to any of these three attractors. For $r \in (13.926, 24.06)$, the clean system has two stable fixed point attractors, C_+ and C_-, and metastable chaos. The latter refers to the phenomenon that for certain initial conditions, the system, especially for r close to 24.06, exhibits chaos-like behavior for a very long period of time before it settles down on either C_+ or C_-. An example for $r = 23.50$ is shown in Fig. 1, where we observe that the metastable chaotic oscillations last more than 1700 natural time units. With a sampling time $\delta t = 0.06$, this amounts to having a time series almost as long as 3×10^4 points for the metastable chaos before the solution leads to C_+. Below, we shall primarily focus our attention on the parameter region leading to metastable chaos.

We only study transient-free asymptotic dynamics of the system. We find that when the noise

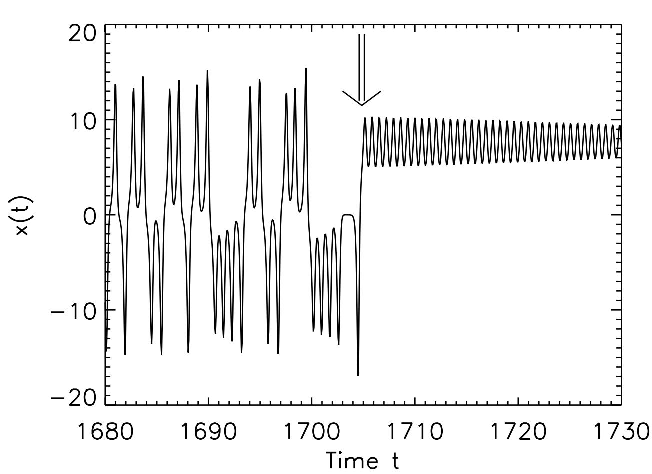


Fig. 1. An example of the metastable Lorenz chaotic signal. An arrow is drawn to separate the metastable and the decaying parts of the signal.

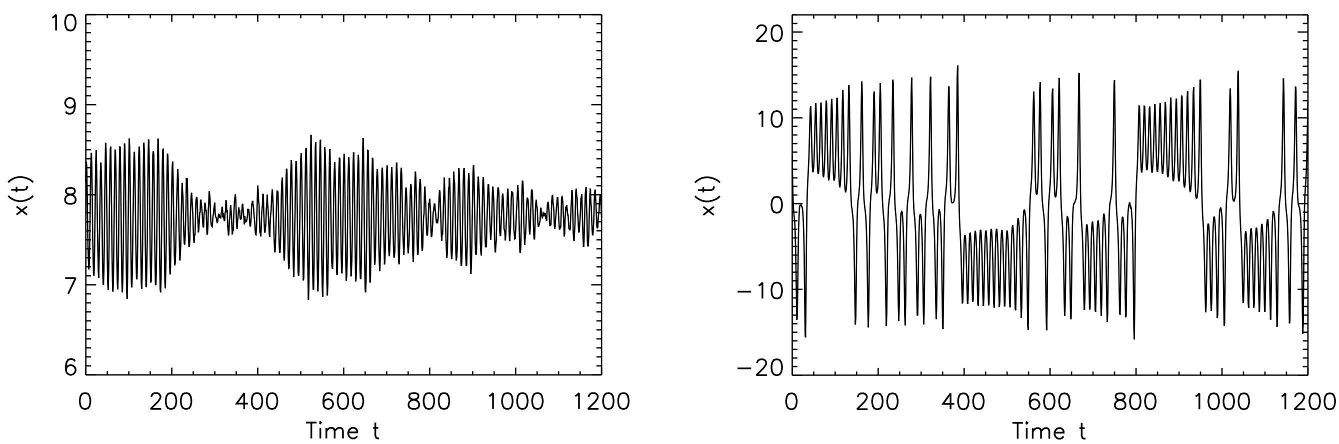


Fig. 2. Time series for the Lorenz equations with $r = 23.60$, (a) $D = 0.2$ and (b) $D = 1.0$.

level is low, the motion is oscillatory with small amplitude. An example for $r = 23.60$ and $D = 0.2$ is shown in Fig. 2(a). The parameter region admitting such motions is indicated in Fig. 3 as region I. With the same level of noise, the amplitude of the oscillatory motions is smaller when the bifurcation parameter r is decreased. Note that the period of the time series of Fig. 2(a) is fairly well defined. This can be readily seen from the power spectral density of the time series, as shown in Fig. 4, where we observe a sharp peak around $f_0 = 1.5\,\text{Hz}$ as well as its second harmonic. However, the amplitude of the time series varies wildly, with the time scale of the envelope variation much longer than the mean oscillation period. This feature is very similar to that observed by [Juel *et al.*, 1997] when studying the effects of noise on the Hopf bifurcations. However, this temporal behavior is not characterized by a single time scale, since we do not observe any specific spectral peaks for $f < f_0$. This suggests that such low-frequency variations may be characterized as fractal variations. This is indeed so, as we shall elaborate in the next section.

What happens when the noise level D is increased? When the noise level is high enough and r is not much smaller than 24.06, the motion becomes chaos-like. An example for $r = 23.60$ and $D = 1.0$ is shown in Fig. 2(b). The parameter region admitting such motions is indicated in Fig. 3 as region II. This chaos-like motion manifests

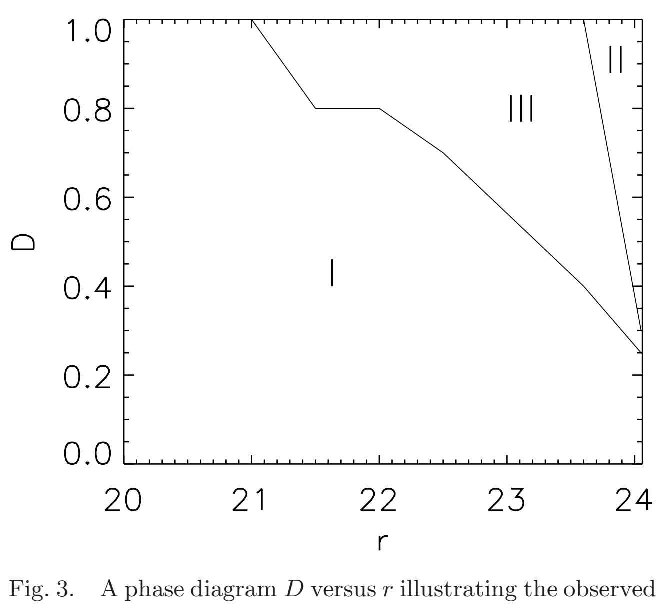


Fig. 3. A phase diagram D versus r illustrating the observed asymptotic dynamics of the noisy Lorenz system for $r \leq 24.05$. Region I: noisy dynamics around the two fixed point solutions; Region II: noise-induced chaos; Region III: intermittency.

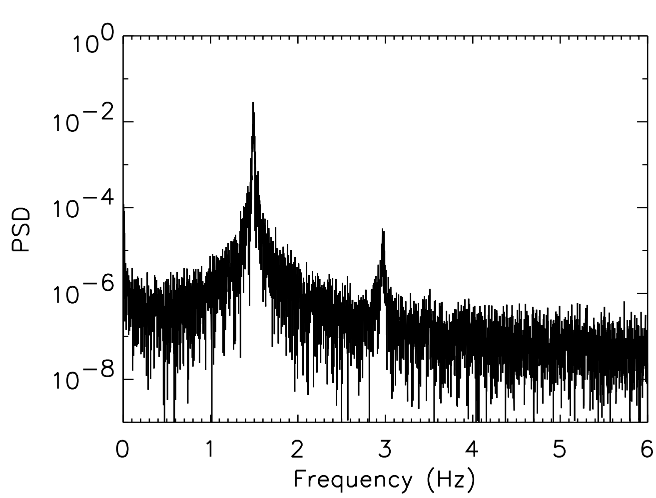


Fig. 4. The power spectral density (PSD) for the time series of Fig. 2(a).

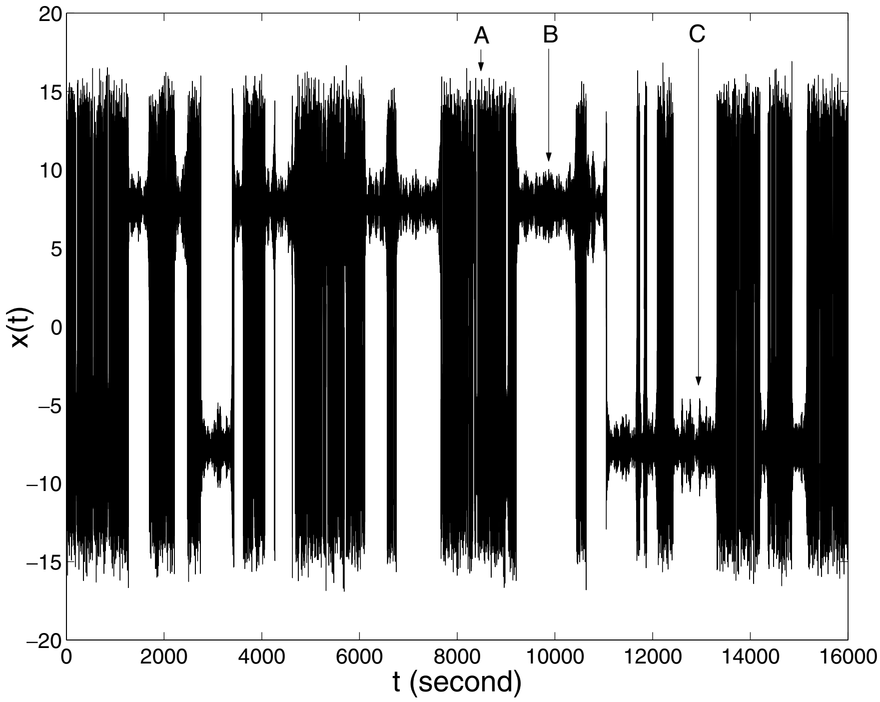


Fig. 5. Intermittency data for $r = 23.60$ and $D = 0.5$. When enlarged, data segment A looks like that shown in Fig. 2(b), data segment B, being around the fixed point C_+, looks like that shown in Fig. 2(a), and data segment C is around the fixed point C_-.

the existence of the metastable chaos for the clean system. When the bifurcation parameter r is smaller than 23.60, more noise is needed to induce and sustain the metastable chaos. However, when r is close to 24.06, say, 24.05, in order to induce and sustain the metastable chaos, the noise level D can be much lower. For example, $D = 0.3$ for $r = 24.05$.

What happens if r is slightly larger than 24.06? Since the clean system now has three attractors, C_+, C_-, and a strange attractor, one may expect that C_+ and C_- are annihilated when noise exceeds certain threshold value, leaving the strange attractor the only observable. This is indeed so. However, a little surprisingly, the threshold for the noise level does not differ much when r varies slightly above 24.06. For example, we have found that $D = 0.2$ for both $r = 24.07$ and 24.10. Thus, 24.06 no longer seems to be a bifurcation point when there is noise in the system. In the next section, we shall examine whether such noise-induced and sustained chaos-like motions can be characterized as low-dimensional chaos or not.

For the noise level lower than that sufficient to induce and sustain metastable chaos, we have the more interesting intermittent behavior. The parameter region admitting intermittency is indicated in Fig. 3 as region III. Figure 5 shows an example for $r = 23.60$ and $D = 0.7$. We observe that the system randomly switches between one of the two fixed point attractors and the metastable chaos.

3. Characterization of Noise-Induced Oscillatory and Chaos-like Motions

In order for the paper to be directly relevant to experimental data analysis, in this section, we assume what can be measured is a scalar time series, $x(1), x(2), \ldots, x(N)$. For convenience, we shall first normalize the scalar time series into the unit interval $[0, 1]$. We then use the time delay embedding procedure [Packard *et al.*, 1980; Takens, 1981; Sauer *et al.*, 1991] to construct vectors of the form: $X_i = [x(i), x(i+L), \ldots, x(i+(m-1)L)]$, with m being the embedding dimension and L the delay time chosen according to optimization criterion [Gao & Zheng, 1993, 1994b]. For ease of exposition, we first characterize noise-sustained metastable chaos (region II in Fig. 3), then characterize noisy oscillatory motions (region I in Fig. 3), and finally, characterize intermittency (region III in Fig. 3).

3.1. *Noise-sustained metastable chaos*

We wish to determine whether they can be characterized as truly chaotic or mostly dominated by noise. We employ the time-dependent exponent curves [Gao & Zheng, 1993, 1994a, 1994b]. This is one of the more stringent tests for chaos, and has found applications in the study of the effects of noise on dynamical systems [Gao *et al.*, 1999a; Gao *et al.*, 1999b] and experimental time series [Bandyopadhyay & Sood, 2001; Bandyopadhyay *et al.*, 2000; Venkadesan *et al.*, 2000]. A new multiscale complexity measure — the scale-dependent Lyapunov exponent [Gao *et al.*, 2006b; Gao *et al.*, 2008] — is based on the derivative of the time-dependent exponent curves; this new measure is able to classify all major types of complex motions, including truly low-dimensional chaos, noisy chaos, noise-induced chaos, random $1/f^{\alpha}$ and α-stable Levy processes, stochastic oscillations, and complex motions with chaotic behavior on small scales but diffusive behavior on large scales.

The time-dependent exponent $\Lambda(k)$ curves are defined as [Gao & Zheng, 1993, 1994a, 1994b],

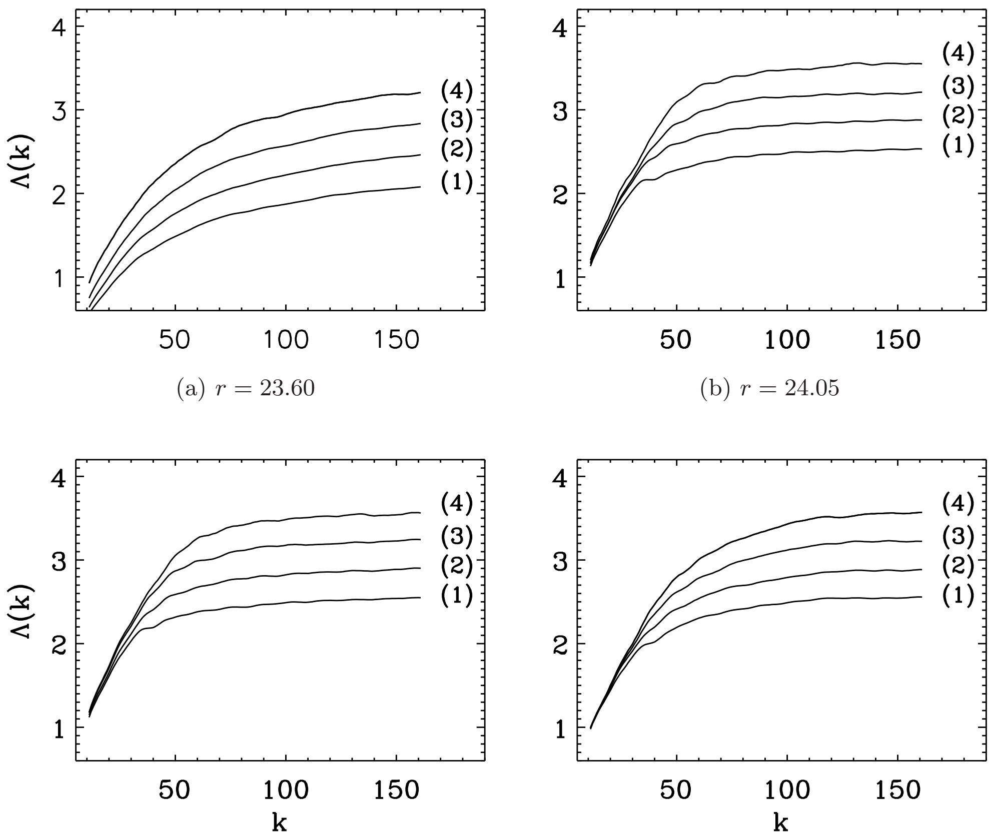
$$\Lambda(k) = \left\langle \ln\left(\frac{\|X_{i+k} - X_{j+k}\|}{\|X_i - X_j\|}\right)\right\rangle \tag{2}$$

with $d \leq \|X_i - X_j\| \leq d + \Delta d$, where d and Δd are prescribed small distances, and the norm of the vectors, $\|X_i\|$, is Euclidean. The angle brackets denote ensemble averages of all possible pairs of (X_i, X_j). The integer k, called the evolution time, corresponds to time $k\delta t$, where δt is the sampling time. Note that geometrically $(d, d + \Delta d)$ defines a shell, and a shell captures the notion of scale. The computation is carried out for a series of shells. For clean chaotic systems, the $\Lambda(k)$ curves first increase linearly with k until some predictable time scale, k_p, then flattens [Gao *et al.*, 1999b]. The linearly increasing parts of the $\Lambda(k)$ curves corresponding to different shells collapse together to form a common envelope, the slope of which estimates the largest positive Lyapunov exponent. This property forms a direct dynamical test for deterministic chaos [Gao & Zheng, 1993, 1994a, 1994b], since the existence of the common envelope guarantees that a robust positive Lyapunov exponent will be obtained by different researchers no matter which shell they use in the computation, thus ensuring determinism. For noisy chaotic systems, the linearly increasing parts of the $\Lambda(k)$ curves correspond to small shells breaking away from the envelope. The stronger the noise, the more the $\Lambda(k)$ curves break away from the envelope [Gao, 1997]. In short, when the dynamics is dominated by noise, then the common envelope is absent. Note that conventional methods for calculating the Lyapunov exponent, such as the one developed by [Wolf *et al.*, 1985], amount to computing $\Lambda(k)$ for $d < d_0$, where d_0 is a small distance selected more or less arbitrarily, then obtaining $\Lambda(k)/k$, for not too large k, as an estimate of the largest Lyapunov exponent. It is now clear that when the common envelope is absent, then the estimated values for the largest Lyapunov exponent are not comparable among different researchers, since d_0 is arbitrary. This is a form of randomness!

Let us now examine the $\Lambda(k)$ curves for the time series of Fig. 2(b). It is shown in Fig. 6(a). We observe that no common envelope exists for this case. Hence, we conclude that the chaos-like signal shown in Fig. 2(b) is dominated by noise.

Does the above result imply that all the noise-sustained metastable chaos for $r < 24.06$ are basically noisy motions? To find the answer, we have analyzed the case for $r = 24.05$. Its $\Lambda(k)$ curves are shown in Fig. 6(b). For comparison, we have also analyzed the two cases of $r = 24.07$ and 24.10. Recall that when $r > 24.06$, the system has two stable fixed point solutions, C_+ and C_-, and a strange attractor. To ensure that originally the system is not running on the strange attractor, we integrate the Lorenz equations with initial conditions to be on either C_+ or C_-. We find that when the noise level is low, the motion is oscillatory, similar to that depicted in Fig. 2(a). When the noise is strong enough, C_+ and C_- are annihilated, leaving the strange attractor the only observable. We choose the lowest noise level that ensures such behavior. The $\Lambda(k)$ curves for noise-induced chaotic signals, for $D = 0.2, r = 24.07$ and $D = 0.2, r = 24.10$, are shown in Figs. 6(c) and 6(d). From Figs. 6(b)–6(d) we observe that when r is very close to 24.06, the common envelope is fairly well defined. Hence, we conclude that those signals are mostly chaotic. We also note that among the three cases depicted in Figs. 6(b)–6(d), the envelope is best defined for $r = 24.10$. Hence, further above the bifurcation point $r = 24.06$, the noise-induced chaos is better defined.

We noted above that $r = 24.06$ no longer seems to be a bifurcation point when the system has noise. Here we observe quantitatively that when the noise level is suitable, the actual bifurcation occurs slightly ahead of the critical point of $r = 24.06$. This is a situation similar to the Hopf bifurcation with noise, as observed by [Hoffman, 1982] and [Juel *et al.*, 1997].



(a) $r = 23.60$ (b) $r = 24.05$

(c) $r = 24.07$ (d) $r = 24.10$

Fig. 6. The $\Lambda(k)$ curves for the time series generated from the noise-driven Lorenz equations with (a) $r = 23.60$ and $D = 1.0$, (b) $r = 24.05$ and $D = 0.3$, (c) $r = 24.07$ and $D = 0.2$, and (d) $r = 24.10$ and $D = 0.2$. The numbers 1 to 4 correspond to shells defined by $(2^{-(i+1)/2}, 2^{-i/2})$ with $i = 9$ to 12. The embedding parameters are $m = 4$ and $L = 3$, and the sampling time is $\delta t = 0.06$.

In summary, for region II of Fig. 3, the chaotic motion is better defined when noise is small and r is increased. When noise is strong, even though the signals appear like chaos, they are dominated by noise and not truly characterized by exponential divergence.

3.2. *Noise-induced oscillatory motions*

We now consider the oscillatory signals depicted in Fig. 2(a). As expected, the $\Lambda(k)$ curves of such signals for k not too large do not form an envelope, thus, such signals are noisy but not chaotic. While interesting, such a conclusion is not very instructive. We ask — Can we find any scaling laws for this kind of signals?

Encouraged by our earlier studies of simulated stochastic oscillators and experimental oscillatory data [Gao, 1997; Gao *et al.*, 1999b; Gao & Tung, 2002], we compute $\langle \ln \|X_{i+k} - X_{j+k}\| \rangle$ based on Eq. (2), for k large compared to the mean period of the oscillation, and check whether there is a scaling law between $\langle \ln \|X_{i+k} - X_{j+k}\| \rangle$ and $\ln k^{\alpha}$, where $\alpha > 0$ is called the diffusion exponent if a scaling law does exist. In other words, we examine if the long-term divergence between two nearby orbits follows

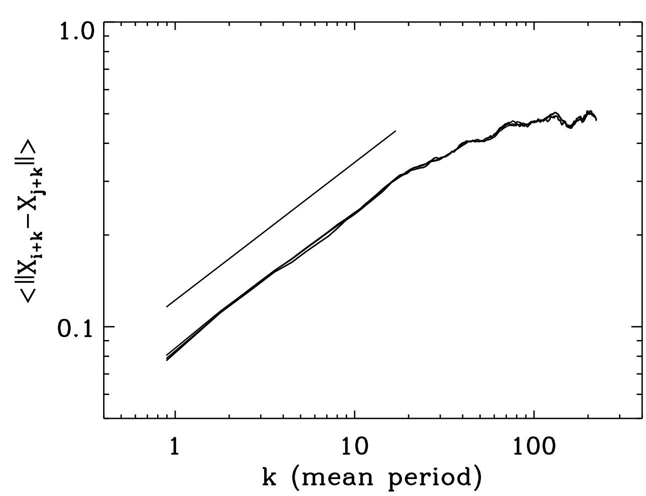


Fig. 7. Log-log plot of the displacement curves for the time series of Fig. 2(a).

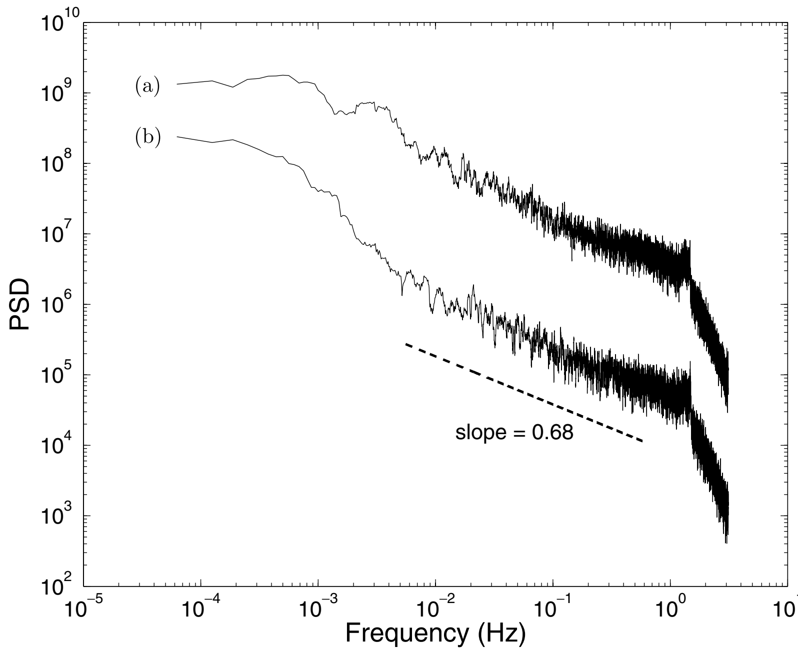


Fig. 8. The PSD for the intermittency signals of $r = 23.60$ and two noise levels, (a) $D = 0.7$, and (b) $D = 0.5$. For better illustration, curve (b) is shifted downward by two decades.

power-law relations. Figure 7 shows the log-log plot of $\langle \|X_{i+k} - X_{j+k}\| \rangle$ versus k, where k is in units of the mean oscillation period. We indeed observe a very well-defined power-law scaling relation, where the scaling regime extends from 1 to more than 10 oscillation periods. Via a linear fit, as shown by the straight line slightly above the $\langle \|X_{i+k} - X_{j+k}\| \rangle$ versus k curve, we find that the diffusion exponent $\alpha = 0.45$. In [Gao *et al.*, 1999b], a diffusion with an exponent of 0.5 is termed normal diffusion, similar to the standard Brownian motions. Since here the exponent is slightly less than 0.5, the diffusional motion slightly deviates from the normal type.

3.3. *Intermittency*

We first describe the general behavior of the system. When r is fixed and noise is small, the system stays around either C_+ or C_- for a long period of time before switching to the chaotic state, where it can only stay for a very brief period. When noise is increased, the rate of switching is increased, as well as the time taken by the system to stay in the chaotic state. When noise is sufficiently large, chaos-like motion dominates, and again, transitions from chaos-like state to either of the fixed point solutions become rare. When the transitions are rare, if one can only observe the dynamics of the system for a limited amount of time, then it may suffice to know the characteristics of the dynamics for each phase, as we have done in parts A and B of this section. When the transitions are fairly frequent and the system spends comparable amount of time in individual phases, such as shown in Fig. 5, a reasonably long time series would already provide valuable information on the characteristics of the transitions. One interesting characteristic is the $1/f$-type behavior shown in Fig. 8, where the parameters are (a) $r = 23.60, D = 0.7$, and (b) $r = 23.60, D = 0.5$. Such $1/f$-type behavior may be better characterized by random fractal theory [Gao *et al.*, 2006c; Gao *et al.*, 2008]. We shall not further pursue it here, however.

4. Conclusions

To gain insights into the effects of noise on multistability, we studied the Lorenz equations with stochastic forcing, focusing on the parameter range in and slightly above the metastable chaos regime. In the metastable chaos regime, the clean system admits two stable fixed point attractors, two unstable periodic solutions, and metastable chaos, i.e. a long chaos-like transience. Above the metastable chaos regime, the clean system admits two stable fixed point attractors, two unstable periodic solutions, and a strange attractor. With stochastic forcing, we observed three interesting behaviors: (i) noise induces oscillatory motions with well-defined periods, a phenomenon similar to stochastic resonance but without a weak periodic forcing; (ii) noise annihilates the two stable fixed point solutions, leaving the originally transient metastable chaos the only observable; and (iii) noise induces hopping between one of the fixed point solutions and the metastable chaos, thus a three-state intermittency phenomenon.

We have also shown that the noise-induced oscillatory motions can be characterized by diffusional scaling laws. Thus, the low-frequency variations in the amplitude of the noise-induced oscillatory motions are fractal variations without a characteristic time scale. Such low-frequency variations were also reported, but not characterized by [Juel *et al.*, 1997], when studying the effects of noise on the Hopf bifurcations. When the parameter value is away from the bifurcation point, the noise-induced and sustained metastable chaos cannot be characterized as chaotic, but are mostly noise. However, when the parameter value is near the bifurcation point, the noise-sustained metastable chaotic signals enable objective computation of the largest positive Lyapunov exponent, and thus qualify as true chaotic signals, characterized by exponential divergence between nearby trajectories in the phase space. Since motions around either of the fixed point solutions are characterized by stochastic oscillations (Fig. 2(a) and Fig. 4) with a power-law divergence (Fig. 7), the intermittency phenomenon may be viewed as a switching between power-law sensitivity to initial conditions and exponential sensitivity to initial conditions [Gao *et al.*, 2005].

While we have found that the intermittency phenomenon behaves as a $1/f$-type process when the transitions are fairly frequent, it would be very desirable to determine how long the system can dwell in each of the intermittent states. It would also be interesting to consider the effect of noise on the global structure of attractors in phase space. This amounts to fully characterizing not only the basins of attraction for all the attractors but also how they shrink/enlarge as noise and r are changed. We shall pursue these topics in future.

The rich multistable behaviors and intermittency induced by noise in the Lorenz system have an interesting implication: noise can act as a controlling parameter, to control transition rate when a system is in an intermittency state. A similar result in a very different system is found in the genetic switch of [Hasty *et al.*, 2000], which uses noise to control transitions in a bistable genetic system. It would be interesting to systematically explore the capability of this idea in physical systems, such as lasers [Pisarchik & Kuntsevich, 2002; Martinez-Zerega *et al.*, 2003].

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